A NONADAPTED STOCHASTIC CALCULUS AND NON STATIONARY EVOLUTION IN FOCK SCALE

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Abstract. A generalized definition of quantum stochastic (QS) integrals and differentials is given in the free of adaptiveness and dimensionality form in terms of Malliavin derivative on a projective Fock space, and their uniform continuity with respect to the inductive limite convergence is proved. A new form of QS calculus based on an inductive $\star$-algebraic structure in an indefinite space is developed and a nonadaptive generalization of the QS Itô formula for its representation in Fock space is derived. The problem of solution of general QS evolution equations in a Hilbert space is solved in terms of the constructed operator representation of chronological products, defined in the indefinite space, and isometry and $\star$-homomorphism property respectively for operators and maps of these solutions, corresponding to the peseudounitary and $\star$-homomorphism property of the QS integrable generators is proved.

1. Introduction

The noncommutative generalization of Itô stochastic calculus developed in [1–6], gives an adequate instrument of studying of the behavior of open quantum dynamical systems in a singular coupling with Bose stochastic fields. The quantum stochastic (QS) calculus enables us to solve the old problem of the stochastic description of continuous collapse of the quantum system under a continuous observation by using the stochastic theory of quantum nondemolition measurements and filtering theory [7–9]. This gives the examples of the stochastic nonunitary, nonstationary and even nonadapted evolution equations in Hilbert space, the solution of which requires one to define the chronologically ordered stochastic exponents of operators and maps in an appropriate way.

Here we solve this general problem in the framework of a new QS calculus in Fock space, based on the explicit definition of the QS integrals free of the adaptedness restriction in a uniform inductive topology, given in [10]. We derive the general (nonadapted) Itô formula as a differential of the Wick formula for the normal ordered products, represented in an inductive $\star$-algebra with respect to an indefinite metric structure. The QS generalization of Itô formula for adapted processes was obtained by Hudson and Parhasarathy in [1], where the unitary QS evolution was constructed for the case of time and field independent QS generators L. They used the QS integral for an adapted operator-valued function D_t as the limit of Itô integral sums in the weak operator topology, defined as in classical case due to commutativity of forward QS differentials $d\Lambda(t) = \Lambda(t+dt) - \Lambda(t)$ with D_t. In this

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approach the QS evolution for nonstationary generating operators of QS differential equations was obtained for some finite dimensional cases by Holevo [11].

An another definition of QS integrals, based on the Berezin-Bargman calculus in terms of kernels of operators in Fock space was proposed by Maassen [3]. One can show that Maassen kernel calculus corresponds to the particular cases of our QS calculus, which is given directly in terms of the Fock representation of integrated operators, instead of kernels [3,4]. Using this new calculus we construct also the explicit solution of the nonstationary, non Markovian, even nonadapted QS Langevin equations for a QS differentiable stochastic process in the sense [12,13] over a unital $\star$-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ as the Fock representation of an recursively defined operator-valued process in a pseudo Hilbert space with noninner QS-integrable generators. Such QS evolution in a Markovian stationary case was constructed recently by Evans and Hudson [14] and in a nonstationary case by Lindsay and Parthasarathy [15]. We shall obtain the existence and uniqueness of Evans-Hudson flow in free dimensional Markovian case by the estimating of the explicit solution in the introduced inductive uniform topology under the natural integrability conditions of time dependent structural coefficients.

2. Nonadapted QS integrals and differentials

Let $\mathcal{H}$ be a Hilbert space with probability vectors $h \in \mathcal{H}$, $\|h\| = 1$, of a quantum dynamical object, described at any instant $t \in \mathbb{R}_+$ by the algebra $\mathcal{B}(\mathcal{H})$ of all linear bounded operators L in $\mathcal{H}$ with Hermitian involution $L \to L^*$ and identity operator I. Let X be a Borel space with a positive measure dx, say $X = \mathbb{R}_+ \times \mathbb{R}^d$, and let $\{\mathcal{E}(x), x \in X\}$ be a family of complex Euclidean (Hilbert) spaces $\mathcal{E}(x)$, describing the quantum field (noise) at a point $x \in X$ of a dimensionality $\dim \mathcal{E}(x) \leq \infty$, usually identified with a space $\mathcal{E}$. We denote by $\mathcal{K} = \int^{\oplus} \mathcal{E}(x) dx$ the Hilbert integral of the field state spaces $\mathcal{E}(x)$, that is the space of all square integrable vector-functions $k\colon x \to k(x) \in \mathcal{E}(x)$,

$$\langle k|k\rangle = \int \|k(x)\|^2 dx < \infty \ , \ \|k(x)\|^2 = \langle k|k\rangle(x),$$

and by $\Gamma(\mathcal{K})$ the Fock space of symmetrical tensor-functions $k(x_1, \dots, x_n)$, $n = 0, 1, \dots$, with values in $\mathcal{E}(x_1) \otimes \dots \otimes \mathcal{E}(x_n)$. Let us assume the absolute continuity $\mathrm{d}x = \lambda(t, \mathrm{d}x)\mathrm{d}t$ with respect to a measurable map $t\colon X \to \mathbb{R}_+$, say $t(x) = t$, $\lambda(t, \mathrm{d}x) = \mathrm{d}\mathbf{x}$ for $x = (t, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^d$, such that

$$\int_\Delta f(t(x))\mathrm{d}x = \int_0^\infty f(t)\lambda(t, \Delta)\mathrm{d}t$$

for any integrable $\Delta \subseteq X$ and essentially bounded function $f\colon \mathbb{R}_+ \to \mathbb{C}$. Then one can represent the Fock space $\Gamma(\mathcal{K})$ as the Hilbert integral $\mathcal{F} = \int_{\mathcal{X}}^{\oplus} \mathcal{E}^{\otimes}(\varkappa)\mathrm{d}\varkappa$ of the functions

$$k\colon \varkappa \to k(\varkappa) \in \mathcal{E}^{\otimes}(\varkappa), \mathcal{E}^{\otimes}(\varkappa) = \otimes_{x\in\varkappa}\mathcal{E}(x)$$

over the set $\mathcal{X}$ of all finite chains $\varkappa = (x_1, \dots, x_n)$, identified with the indexed subsets $\{x_1, \dots, x_n\} \subset X$ of cardinality $|\varkappa| = n < \infty$ and $\mathrm{d}\varkappa = \prod_{x\in\varkappa} \mathrm{d}x$ under the order $t(x_1) < \dots < t(x_n)$. We shall denote by $t(\varkappa)$ the chains (subsets) $\{t(x)|x \in \varkappa\}$, $\emptyset \in \mathcal{X}$ denotes the empty chain and $1_\emptyset \in \mathcal{F}$ denotes the vacuum function: $1_\emptyset(\varkappa) = 0$, if $\varkappa \neq \emptyset$; $1_\emptyset(\emptyset) = 1$.

This can be done as in the case $X = \mathbb{R}_+, t(x) = x$ by the isometry

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_{X^n} \|k(\varkappa)\|^2 \mathrm{d}\varkappa = \sum_{n=0}^{\infty} \int \cdots \int_{t_1 < \cdots < t_n} \|k(x_1, \ldots, x_n)\|^2 \mathrm{d}x_1 \ldots \mathrm{d}x_n,$$

where the integrals in right hand side is taken over all $\varkappa = \{x_1 < \cdots < x_n\}$ with different $t_i = t(x_i)$ due to

$$\frac{1}{n!} \int_0^\infty \ldots \int_0^\infty f(t_1, \ldots, t_n) \mathrm{d}t_1 \cdots \mathrm{d}t_n = \int_0^\infty \mathrm{d}t_1 \int_{t_1}^\infty \mathrm{d}t_2 \ldots \int_{t_{n-1}}^\infty \mathrm{d}t_n f(t_1, t_2, \ldots, t_n)$$

for the symmetrical function

$$f(t_1, \ldots, t_n) = \int_{X^n} \|k(\varkappa)\|^2 \prod_{i=1}^n \lambda(t_i, \mathrm{d}x_i).$$

One can consider the set X as the space with a casual preorder $\lesssim$ [12], and the increasing map $t\colon x \lesssim x' \Rightarrow t(x) \leq t(x')$ as the local time, if for any $x \in X$ and $t' > t(x)$ there exists $x' \in X$ such that $t(x') = t'$ (As it is for the map $t(x) = t$ with respect to the Galilean or Einsteinian order in space-time $X = \mathbb{R}_+ \times \mathbb{R}^{\mathrm{d}}$).

Let us denote by $\mathcal{F}(\xi) = \int_{\mathcal{X}}^{\oplus} \xi^{|\varkappa|} \mathcal{E}^{\otimes}(\varkappa) \mathrm{d}\varkappa$ for all $\xi > 0$ the Hilbert scale of Fock spaces $\mathcal{F}(\xi) \subseteq \mathcal{F}(\zeta)$, $\xi \geq \zeta$, defined by the scalar products

$$\|k\|^2(\xi) = \sum_{n=0}^{\infty} \xi^n \int \cdots \int_{0 \leq t_1 < \cdots < t_n < \infty} \|k(x_1, \ldots, x_n)\|_\xi^2 \mathrm{d}x_1 \cdots \mathrm{d}x_n \ ,$$

by $\mathcal{G}(\xi) = \mathcal{H} \otimes \mathcal{F}(\xi)$ the Hilbert tensor products, by $\mathcal{G}^+ = \mathcal{G}(\xi^+)$, $\mathcal{G} = \mathcal{G}(1)$, $\mathcal{G}_- = \mathcal{G}(\xi_-)$ the Hilbert subspaces $\mathcal{G}^+ \subseteq \mathcal{G} \subseteq \mathcal{G}_-$ for some $\xi^+ \geq 1 \geq \xi_-$, and let us note that any linear operator $L \in \mathcal{B}(\mathcal{H})$ can be considered as (ξ^+, ξ_-)-continuous (bounded) operator $B : \mathcal{G}^+ \to \mathcal{G}_-$ of the form $B = L \otimes \hat{1}$, where $\hat{1}$ means the identity operator $\hat{1} = \int_{\mathcal{X}}^{\oplus} I^{\otimes}(\varkappa) \mathrm{d}\varkappa \equiv I^{\otimes}$ in $\mathcal{F} = \mathcal{F}(1)$, $I^{\otimes}(\varkappa) = \otimes_{x \in \varkappa} I(x)$, considered as the identical map $\mathcal{F}(\xi^+) \to \mathcal{F}(\xi_-)$. Following [2,8] we define the QS integral $\Lambda^t(\mathbf{D}) = \int_0^t \mathrm{d}\Lambda^s(\mathbf{D})$ for a table $\mathbf{D} = (D^\mu_\nu)^{\mu=-,0}_{\nu=0,+}$ of functions $\{D^\mu_\nu(x), x \in X\}$ with values in continuous operators

$$D^0_0(x) : \mathcal{G}^+ \otimes \mathcal{E}(x) \to \mathcal{G}_- \otimes \mathcal{E}(x) \ , \ D^-_+(x) : \mathcal{G}^+ \to \mathcal{G}_-, \tag{1.1a}$$

$$D^0_+(x) : \mathcal{G}^+ \to \mathcal{G}_- \otimes \mathcal{E}(x) \ , \ \ D^-_0(x) : \mathcal{G}^+ \otimes \mathcal{E}(x) \to \mathcal{G}_- \ , \tag{1.1.b}$$

as the sum $\Lambda^t(\mathbf{D}) = \sum_{\mu,\nu} \Lambda^\nu_\mu(t, D^\mu_\nu)$ of the operators $\Lambda^\nu_\mu(t, D) : a \in \mathcal{G} \mapsto \Lambda^\nu_\mu(t, D)a$, acting as

$$[\Lambda^0_0(t, D^0_0)a](\varkappa) = \textstyle\sum_{x \in \varkappa^t} [D^0_0(x) \dot{a}(x)](\varkappa \backslash x) \tag{1.2a}$$

$$[\Lambda^+_0(t, D^0_+)a](\varkappa) = \textstyle\sum_{x \in \varkappa^t} [D^0_+(x) a](\varkappa \backslash x) \tag{1.2b}$$

$$[\Lambda^0_-(t, D^-_0)a](\varkappa) = \textstyle\int_{X^t} [D^-_0(x) \dot{a}(x)](\varkappa) \mathrm{d}x \tag{1.2c}$$

$$[\Lambda^+_-(t, D^-_+ a](\varkappa) = \textstyle\int_{X^t} [D^-_+(x) a](\varkappa) \mathrm{d}x \ , \tag{1.2d}$$

Here $\varkappa^t = \varkappa \cap X^t$, $X^t = \{x \in X : t(x) < t\}$, $\varkappa \backslash x = \{x' \in \varkappa : x' \neq x\}$, and $a \mapsto \dot{a}(x)$ is the point (Malliavin [16,17]) derivative $\mathcal{G}^+ \to \mathcal{G}^+ \otimes \mathcal{E}(x)$, evaluated in the Fock representation almost everywhere as $[\dot{a}(x)](\varkappa) = a(x \sqcup \varkappa)$, where $x \sqcup \varkappa = \{x, \varkappa : x \notin \varkappa\}$ is the disjoint union of the chains x, $\varkappa \in \mathcal{X}$. The operator-functions (1.2) were defined in [1] as the limits of the QS Itô integral sums with respect to

the gage, creation, annihilation, and time processes respectively for the bounded adapted operator valued functions $D(x) = A(x) \otimes \hat{1}_{[t}$, where $t = t(x), \hat{1}_{[t} = I^{\otimes}_{[t}$ is the identity operator in $\mathcal{F}_{[t} = \int^{\oplus}_{\mathcal{X}_{[t}} \mathcal{E}^{\otimes}(\varkappa)\mathrm{d}\varkappa$, $\mathcal{X}_{[t} = \{\varkappa \in \mathcal{X} | t(\varkappa) \geq t\}$. As it follows from theorem 1 in [9,10] the operators (1.2) are densely defined in $\mathcal{G}$ as (ζ^+, ζ_-)-continuous operators $\mathcal{G}(\zeta^+) \to \mathcal{G}(\zeta_-)$ for any $\zeta^+ > \xi^+$, $\zeta_- < \xi_-$ even for the nonadapted and unbounded D, satisfying local QS-integrability conditions

$$\|D^0_0\|^{\xi_-,t}_{\xi^+,\infty} < \infty, \|D^0_+\|^{\xi_-,t}_{\xi^+,2} < \infty, \|D^-_0\|^{\xi_-,t}_{\xi^+,2} < \infty, \|D^-_+\|^{\xi_-,t}_{\xi^+,1} < \infty \ , \tag{1.3}$$

for all $t \in \mathbb{R}_+$ and some ξ_- , $\xi^+ > 0$, where

$$\|D\|^{\xi_-,t}_{\xi^+,p} = \left(\int_{X^t} \left(\|D(x)\|^{\xi_-}_{\xi^+} \right)^p \mathrm{d}x \right)^{1/p}, \quad \|D\|^{\xi_-}_{\xi^+} = \sup\{\|D\mathbf{a}\|(\xi_-)/\|\mathbf{a}\|(\xi^+)\}.$$

Let us now define the multiple QS integral

$$\Lambda_{[0,t)}(B) = \sum_{n=0}^{\infty} \int \cdots \int_{0 \leq t_1 < \cdots < t_n < t} \mathrm{d}\Lambda^{t_1,\ldots,t_n}(B) \equiv \int_{0 \leq \tau < t} \mathrm{d}\Lambda^{\tau}(B)$$

for the operator-valued function $B(\boldsymbol{\vartheta})$ on the table $\boldsymbol{\vartheta} = (\vartheta^\mu_\nu)^{\mu=-,0}_{\nu=0,+}$ of four subsets $\vartheta^\mu_\nu \in \mathcal{X}$ with values

$$B \begin{pmatrix} \vartheta^-_0 & \vartheta^-_+ \\ \vartheta^0_0 & \vartheta^0_+ \end{pmatrix} : \mathcal{G}(\eta^+) \otimes \mathcal{E}^{\otimes}(\vartheta^-_0) \otimes \mathcal{E}^{\otimes}(\vartheta^0_0) \to \mathcal{G}(\eta_-) \otimes \mathcal{E}^{\otimes}(\vartheta^0_0) \otimes \mathcal{E}^{\otimes}(\vartheta^0_+) \tag{1.4}$$

as the operators in $\mathcal{G}$ with the action

$$[\Lambda_{[0,t)}(B)a](\varkappa) = \sum_{\vartheta^0_0 \sqcup \vartheta^0_+ \subseteq \varkappa^t} \int_{\mathcal{X}^t} \int_{\mathcal{X}^t} [B(\boldsymbol{\vartheta})\dot{a}(\vartheta^-_0 \sqcup \vartheta^0_0)](\vartheta^0_-)\mathrm{d}\vartheta^-_0 \mathrm{d}\vartheta^-_+ \ . \tag{1.5}$$

Here $\vartheta^0_- = \varkappa \cap \overline{(\vartheta^0_0 \sqcup \vartheta^0_+)} = \varkappa \backslash \vartheta^0_0 \backslash \vartheta^0_+$ is the difference of a subset $\varkappa \subset X$ and the partition $\vartheta^0_0 \sqcup \vartheta^0_+$ as the disjoint union $\vartheta^0_0 \bigcup \vartheta^0_+ \subseteq \varkappa$, $\vartheta^0_0 \cap \vartheta^0_+ = \emptyset$, $\mathcal{X}^t = \{\varkappa \in \mathcal{X} | \varkappa \subset X^t\}$, and the point (Malliavin [16]) derivative

$$\dot{a}(\vartheta) = \int^{\oplus} a(\varkappa \sqcup \vartheta)\mathrm{d}\varkappa \in \mathcal{G}^+ \otimes \mathcal{E}^{\otimes}(\vartheta)$$

is defined for almost all $\varkappa \in \mathcal{X}$, $\varkappa \cap \vartheta = \emptyset$ as $\dot{a}(\varkappa, \vartheta) = a(\varkappa \sqcup \vartheta)$ by a vector-function $a \in \mathcal{G}^+$. We shall say that the function B is locally QS integrable (in a uniform inductive limit), if for any $t \in \mathbb{R}_+$ there exists a pair $(\eta^\bullet, \eta_\bullet)$ of triples $\eta^\bullet = (\eta^-, \eta^0, \eta^+)$, $\eta_\bullet = (\eta_-, \eta_0, \eta_+)$ of numbers $\eta^\mu > 0$, $\eta_\nu > 0$, for which $\|B\|^{\eta_\bullet}_{\eta^\bullet}(t) < \infty$, where

$$\|B\|^{\eta_\bullet}_{\eta^\bullet}(t) = \int_{\mathcal{X}^t} \mathrm{d}\vartheta^-_+ \left(\int_{\mathcal{X}^t} \int_{\mathcal{X}^t} \frac{(\eta_+)^{|\vartheta^0_+|}}{(\eta^-)^{|\vartheta^-_0|}} \sup_{\mathcal{X}^t} \frac{(\eta_0)^{|\vartheta^0_0|}}{(\eta^0)^{|\vartheta^0_0|}} \left(\|B(\boldsymbol{\vartheta})\|^{\eta_-}_{\eta^+} \right)^2 \mathrm{d}\vartheta^0_+ \mathrm{d}\vartheta^-_0 \right)^{1/2} \tag{1.6}$$

(sup is taken as essential supremum over $\vartheta^0_0 \in \mathcal{X}^t$). As it follows from the next theorem, the function $B(\boldsymbol{\vartheta})$ in QS integral (1.5) can be defined up to the equivalence having the kernel $B \approx 0 \Leftrightarrow \|B\|^{\eta_\bullet}_{\eta^\bullet}(t) = 0$ for all $\eta_\bullet, \eta^\bullet$ and t. In particular, one can define it only for the tables $\boldsymbol{\vartheta} = (\vartheta^\mu_\nu)$, which are partitions $\varkappa = \sqcup \vartheta^\mu_\nu$ of the

chains $\varkappa \in \mathcal{X}$, i.e. for $\boldsymbol{\vartheta} = \sqcup_{x\in\varkappa}\mathbf{x}$, where $\mathbf{x}$ means one from the four single point (elementary) tables

$$\mathbf{x}_0^0 = \begin{pmatrix} \emptyset & \emptyset \\ x & \emptyset \end{pmatrix},\ \mathbf{x}_+^0 = \begin{pmatrix} \emptyset & \emptyset \\ \emptyset & x \end{pmatrix},\ \mathbf{x}_0^- = \begin{pmatrix} x, & \emptyset \\ \emptyset & \emptyset \end{pmatrix},\ \mathbf{x}_+^- = \begin{pmatrix} \emptyset & x \\ \emptyset & \emptyset \end{pmatrix}\ .$$

Theorem 1. *If B is a locally QS integrable function (1.4), then the multiple integral (1.5) is a (ξ^+,ξ_-) continuous operator $U^t : \mathcal{G}(\xi^+) \to \mathcal{G}(\xi_-)$ for $\xi^+ \geq \sum_\mu \eta^\mu, \xi_-^{-1} \geq \sum_\nu \eta_\nu^{-1}$, having the estimate*

$$\|\Lambda_{[0,t)}(B)a\|(\xi_-) \leq\ \|B\|_{\eta\bullet}^{\eta\bullet}\|a\|(\xi^+)\ ,\ \forall a \in \mathcal{G}(\xi^+)\ .$$

The formally conjugated in $\mathcal{G}$ operator is defined as QS integral

$$\Lambda_{[0,t)}(B)^* = \Lambda_{[0,t)}(B^\star), B^\star(\boldsymbol{\vartheta}) = B(\boldsymbol{\vartheta}^\star)^*, \boldsymbol{\vartheta}^\star = \begin{pmatrix} \vartheta_+^0 & \vartheta_+^- \\ \vartheta_0^0 & \vartheta_0^- \end{pmatrix}\ , \tag{1.7}$$

which is the continuous operator $\mathcal{G}(1/\xi_-) \to \mathcal{G}(1/\xi^+)$ with

$$\|\Lambda_{[0,t)}(B^\star)\|_{1/\xi_-}^{1/\xi^+} = \|\Lambda_{[0,t)}(B)\|_{\xi^+}^{\xi_-} \equiv \sup\{\|U^t a\|(\xi_-)/\|a\|(\xi^+)\}\ .$$

The QS process $U^t = \Lambda_{[0,t)}(B)$ has a QS differential $\mathrm{d}U^t = \mathrm{d}\Lambda^t(\mathbf{D})$ in the sense

$$\Lambda_{[0,t)}(B) = B(\boldsymbol{\emptyset}) + \Lambda^t(\mathbf{D})\ ,\ D_\nu^\mu(x) = \Lambda_{[0,t(x))}(\dot{B}(\mathbf{x}_\nu^\mu))$$

with (ξ^+,ξ_-)-continuous QS derivatives $\mathbf{D} = (D_\nu^\mu)$, densely defined as in (1.5) by $\dot{B}(\mathbf{x},\boldsymbol{\vartheta}) = B(\mathbf{x}\sqcup\boldsymbol{\vartheta})$ for almost all $\boldsymbol{\vartheta} = (\vartheta_\nu^\mu)$, where $\mathbf{x} = (\varkappa_\nu^\mu)$ is one from the elementary tables $\mathbf{x}_\nu^\mu$, $\mu \neq +$, $\nu \neq -$ with $\varkappa_\nu^\mu = x$, and $\vartheta_\nu^\mu \in \mathcal{X}^{t(x)}$.

Let $B(\boldsymbol{\vartheta})$ be defined for any partition $\varkappa = \sqcup\vartheta_\nu^\mu \in \mathcal{X}$ as the solution $B(\boldsymbol{\vartheta}) = \mathbf{L}^\triangleleft(\boldsymbol{\vartheta}) \odot B(\boldsymbol{\emptyset})$ of the recurrency

$$B(\mathbf{x}\sqcup\boldsymbol{\vartheta}) = L(\mathbf{x}) \odot B(\boldsymbol{\vartheta}), \vartheta_\nu^\mu \in \mathcal{X}^{t(x)}$$

with $B(\boldsymbol{\emptyset}) = T^0 \otimes \hat{1}$, i.e.

$$\dot{B}(\mathbf{x},\boldsymbol{\vartheta}) = (L(\mathbf{x}) \otimes \hat{1}) \cdot B(\boldsymbol{\vartheta})\ ,\ B(\boldsymbol{\emptyset}) = T^0 \otimes \hat{1} \tag{1.8}$$

with a table $\mathbf{L} = (L_\nu^\mu)_{\nu=0,+}^{\mu=-,0}$ of operator-valued functions $L_\nu^\mu(x) = L(\mathbf{x}_\nu^\mu)$,

$$L_0^0(x)\ :\ \mathcal{H}\otimes\mathcal{E}(x) \to \mathcal{H}\otimes\mathcal{E}(x)\ ,\ L_+^-(x) : \mathcal{H} \to \mathcal{H}\ , (1.9a)$$
$$L_+^0(x)\ :\ \mathcal{H} \to \mathcal{H}\otimes\mathcal{E}(x)\ ,\ L_0^-(x) : \mathcal{H}\otimes\mathcal{E}(x) \to \mathcal{H}\ , (1.9b)$$

$L \odot B = (L \otimes \hat{1}) \cdot B$, and

$$B(\boldsymbol{\varkappa}) \cdot B(\boldsymbol{\vartheta}) = (B(\boldsymbol{\varkappa}) \otimes I^\otimes(\vartheta_0^0 \sqcup \vartheta_+^0))(B(\boldsymbol{\vartheta}) \otimes I^\otimes(\varkappa_0^- \sqcup \varkappa_0^0))\ ,$$

where $I^\otimes(\varkappa) = \otimes_{x\in\varkappa} I(x), I(x)$ is the identity operator in $\mathcal{E}(x)$. $((B(\mathbf{x}) \cdot B(\boldsymbol{\vartheta})$ in (1.8) means usual product of operators in $\mathcal{G}$, if $\dim\mathcal{E} = 1$).

Then the process $U^t = \Lambda_{[0,t)}(B)$ satisfies the QS differential equation $\mathrm{d}U^t = \mathrm{d}\Lambda^t(\mathbf{L} \odot U^t)$ in the sense

$$U^t = U^0 + \Lambda^t(\mathbf{L} \odot U^t)\ ,\ (\mathbf{L} \odot U)_\mu^\mu(x) = (L(\mathbf{x}_\nu^\mu) \otimes \hat{1})U^{t(x)}\ . \tag{1.10}$$

Proof. Using the sum-point integral property

$$\int \sum_{\sqcup\vartheta_\nu=\vartheta} f(\vartheta_-,\vartheta_0,\vartheta_+)\mathrm{d}\vartheta = \iiint f(\vartheta_-,\vartheta_0,\vartheta_+)\prod_\nu \mathrm{d}\vartheta_\nu$$

of the multiple sum-point integral, we obtain from definition (1.5) for $a, c \in \mathcal{G}$:

$$\int \langle c(\vartheta)|[U^t a](\vartheta)\rangle \mathrm{d}\vartheta = \int_{\mathcal{X}^t} \mathrm{d}\vartheta_+^- \int_{\mathcal{X}^t} \mathrm{d}\vartheta_0^- \int_{\mathcal{X}^t} \mathrm{d}\vartheta_+^0 \int_{\mathcal{X}^t} \mathrm{d}\vartheta_0^0 \langle \dot{c}(\vartheta_0^0 \sqcup \vartheta_+^0)|B(\boldsymbol{\vartheta})\dot{a}(\vartheta_0^- \sqcup \vartheta_0^0)\rangle =$$

$$\int_{\mathcal{X}^t} \mathrm{d}\vartheta_+^- \int_{\mathcal{X}^t} \mathrm{d}\vartheta_0^- \int_{\mathcal{X}^t} \mathrm{d}\vartheta_+^0 \int_{\mathcal{X}^t} \mathrm{d}\vartheta_0^0 \langle B(\boldsymbol{\vartheta})^* \dot{c}(\vartheta_0^0 \sqcup \vartheta_+^0))|\dot{a}(\vartheta_0^- \sqcup \vartheta_0^0)\rangle = \int \langle [U^{t*}c](\vartheta)|a(\vartheta)\rangle \mathrm{d}\vartheta \ ,$$

that is U^{t*} acts as $\Lambda_{[0,t)}(B^\star)$ in (1.5) with $B^\star(\boldsymbol{\vartheta}) = B(\boldsymbol{\vartheta}^\star)^*$. Moreover this equation gives $\|\Lambda_{[0,t)}(B)\|_\xi^{1/\zeta} = \|\Lambda_{[0,t)}(B^\star)\|_\zeta^{1/\xi}$ as

$$\|U\|_\xi^{1/\zeta} = \sup|\langle c|Ua\rangle|/\|a\|(\xi)\|c\|(\zeta) = \sup|\langle U^* c|a\rangle|/\|c\|(\zeta)\|a\|(\xi) = \|U^*\|_\zeta^{1/\xi} \ .$$

Let us estimate the integral $\langle c|U^t a\rangle$, using the Schwarz inequality

$$\int \|\dot{c}(\vartheta)\|(\eta_-^{-1})\|\dot{a}(\vartheta)\|(\eta_+)(\eta_0/\eta^0)^{|\vartheta|/2}\mathrm{d}\vartheta \le \|\dot{c}\|(\eta_-^{-1}, \eta_0^{-1})\|\dot{a}\|(\eta^+, \eta^0)$$

and the following isometricity property of the multiple derivative:

$$\|\dot{a}\|(\xi, \eta) = \left(\iint \xi^{|\vartheta|}\eta^{|\sigma|}\|a(\vartheta \sqcup \sigma)\|^2 \mathrm{d}\vartheta \mathrm{d}\sigma\right)^{1/2} = \|a\|(\xi + \eta) \ .$$

This gives $|\langle c|U^t a\rangle| = |\int \langle c(\varkappa)|[\imath_{[0,t)}^{\otimes}(B)a](\varkappa)\rangle \mathrm{d}\varkappa \le$

$$\begin{aligned}
&\le \int_{\mathcal{X}^t} \mathrm{d}\vartheta_+^- \int_{\mathcal{X}^t} \mathrm{d}\vartheta_0^- \int_{\mathcal{X}^t} \mathrm{d}\vartheta_+^0 \int_{\mathcal{X}^t} \mathrm{d}\vartheta_0^0 \|\dot{c}(\vartheta_0^0 \sqcup \vartheta_+^0)\|(\eta_-^{-1})| \, \|B(\boldsymbol{\vartheta})\|_{\eta^+}^{\eta_-} \|\dot{a}(\vartheta_0^- \sqcup \vartheta_0^0)\|(\eta^+) \\
&\le \int_{\mathcal{X}^t} \mathrm{d}\vartheta_+^- \int_{\mathcal{X}^t} \mathrm{d}\vartheta_0^- \int_{\mathcal{X}^t} \mathrm{d}\vartheta_+^0 \|\dot{c}(\vartheta_+^0)\|(\eta_-^{-1} + \eta_0^{-1})\|\dot{a}(\vartheta_0^-)\|(\eta^+ + \eta^0)\|B\|_{\eta^+,\eta^0}^{\eta_-,\eta_0}(t, \boldsymbol{\vartheta}\backslash\boldsymbol{\vartheta}_0^0) \\
&\le \|c\|(\sum_{\nu=-}^{+} \eta_\nu^{-1})\|a\|(\sum_{\mu=-}^{+} \eta^\mu) \int_{\mathcal{X}^t} \mathrm{d}\vartheta_+^- \left[\int_{\mathcal{X}^t} \mathrm{d}\vartheta_0^- \int_{\mathcal{X}^t} \mathrm{d}\vartheta_+^0 \frac{(\eta_+)^{|\vartheta_+^0|}}{(\eta^-)^{|\vartheta_0^-|}} \|B\|_{\eta^+,\eta^0}^{\eta_-,\eta_0}(t, \boldsymbol{\vartheta}\backslash\boldsymbol{\vartheta}_0^0)^2\right]^{\frac{1}{2}}
\end{aligned}$$

where $\|B\|_{\eta^+,\eta^0}^{\eta_-,\eta_0}(t, \boldsymbol{\vartheta}\backslash\vartheta_0^0) = \mathrm{esssup}_{\vartheta_0^0 \in \mathcal{X}^t} \frac{(\eta_0)^{|\vartheta_0^0|/2}}{(\eta^0)^{|\vartheta_0^0|/2}} \|B(\boldsymbol{\vartheta})\|_{\eta^+}^{\eta_-}$. Hence

$$|\langle c|U^t a\rangle| \le \|c\|(\xi_-^{-1})\|a\|(\xi^+)\|B\|_{\eta^\bullet}^{\eta_\bullet}(t)$$

for $\xi^+ \ge \sum_\mu \eta^\mu, \xi_-^{-1} \ge \sum_\nu \eta_\nu^{-1}$.

Using the definition (1.5) and the property

$$\int_{\mathcal{X}^t} f(\vartheta)\mathrm{d}\vartheta = f(\emptyset) + \int_{\mathcal{X}^t} \mathrm{d}x \int_{\mathcal{X}^{t(x)}} \dot{f}(x, \vartheta)\mathrm{d}\vartheta \ , \ \dot{f}(x, \vartheta) = f(x \sqcup \vartheta) \ ,$$

one can obtain

$$[(U^t - U^0)a](\varkappa) = [(\imath_{[0,t)}^{\otimes}(B) - B(\boldsymbol{\emptyset}))a](\varkappa) =$$

$$\int_{X^t} \mathrm{d}x \sum_{\vartheta_0^0 \sqcup \vartheta_+^0 \subseteq \varkappa}^{t(\vartheta_\nu^0) < t(x)} \int_{\mathcal{X}^{t(x)}} \mathrm{d}\vartheta_+^- \int_{\mathcal{X}^{t(x)}} \mathrm{d}\vartheta_0^- [\dot{B}(\mathbf{x}_+^-, \boldsymbol{\vartheta})\dot{a}(\vartheta_0^- \sqcup \vartheta_0^0) + \dot{B}(\mathbf{x}_0^-, \boldsymbol{\vartheta})\dot{a}(x \sqcup \vartheta_0^- \sqcup \vartheta_0^0)](\vartheta_-^0)$$

$$+ \sum_{x \in \varkappa^t} \sum_{\vartheta_0^0 \sqcup \vartheta_+^0 \subseteq \varkappa}^{t(\vartheta_\nu^0) < t(x)} \int_{\mathcal{X}^{t(x)}} \mathrm{d}\vartheta_+^- \int_{\mathcal{X}^{t(x)}} \mathrm{d}\vartheta_0^- [\dot{B}(\mathbf{x}_+^0, \boldsymbol{\vartheta})\dot{a}(\vartheta_0^- \sqcup \vartheta_0^0) + \dot{B}(\mathbf{x}_0^0, \boldsymbol{\vartheta})\dot{a}(x \sqcup \vartheta_0^- \sqcup \vartheta_0^0)](\vartheta_-^0)$$

$$= \int_{X^t} \mathrm{d}x[D_+^-(x)a + D_0^-(x)\dot{a}(x)](\varkappa) + \sum_{x \in \varkappa^t} [D_+^0(x)a + D_0^0(x)\dot{a}(x)](\varkappa/x) \ .$$

Hence
$$U^t - U^0 = \Lambda_-^+(t, D_+^-) + \Lambda_-^0(t, D_0^-) + \Lambda_0^+(t, D_+^0) + \Lambda_0^0(t, D_0^0) \ ,$$
where $\Lambda_\mu^\nu(t)$ are the QS integrals (1.2) of operators
$$[D_+^\mu(x)a](\varkappa) = \sum_{\vartheta_0^0 \sqcup \vartheta_+^0 \subseteq \varkappa}^{t(\vartheta_\nu^0)<t(x)} \int_{\mathcal{X}^{t(x)}} \mathrm{d}\vartheta_+^- \int_{\mathcal{X}^{t(x)}} \mathrm{d}\vartheta_0^- [\dot{B}(\mathbf{x}_+^\mu, \boldsymbol{\vartheta})\dot{a}(\vartheta_0^- \sqcup \vartheta_0^0)](\vartheta_-^0),$$
$$[D_0^\mu(x)b](\varkappa) = \sum_{\vartheta_0^0 \sqcup \vartheta_+^0 \subseteq \varkappa}^{t(\vartheta_\nu^0)<t(x)} \int_{\mathcal{X}^{t(x)}} \mathrm{d}\vartheta_+^- \int_{\mathcal{X}^{t(x)}} \mathrm{d}\vartheta_0^- [\dot{B}(\mathbf{x}_0^\mu, \boldsymbol{\vartheta})\dot{b}(\vartheta_0^- \sqcup \vartheta_0^0)](\vartheta_-^0) \ ,$$
for $a \in \mathcal{G}^+$, $b \in \mathcal{G}^+ \otimes \mathcal{E}(x)$. This can be written in terms of (1.5) as $D_\nu^\mu(x) = \imath_{[0,t(x))}^{\otimes}(\dot{B}(\mathbf{x}_\nu^\mu))$. Due to the inequality
$$\|U^t\|_{\xi^+}^{\xi_-} \leq \|B\|_{\eta^\bullet}^{\eta_\bullet}(t), \ \ \xi^+ \geq \sum \eta^\mu, \quad \xi_-^{-1} \geq \sum \eta_\nu^{-1}$$
one obtains $\|D_+^-\|_{\xi^+,1}^{\xi_-,t} \leq \|B\|_{\eta^\bullet}^{\eta_\bullet}(t)$:
$$\int_{X^t} \|D_+^-(x)\|_{\xi^+}^{\xi_-} \mathrm{d}x \leq \int_{X^t} \|\dot{B}_+^-(x)\|_{\eta^\bullet}^{\eta_\bullet}[t(x)]\mathrm{d}x = \int_{X^t} \mathrm{d}x \int_{\mathcal{X}^{t(x)}} \|B_+^-(x \sqcup \vartheta)\|_{\eta^\bullet}^{\eta_\bullet}[t(x)]\mathrm{d}\boldsymbol{\vartheta}$$
$$= \int_{\mathcal{X}^t} \|B_+^-(\mathcal{X})\|_{\eta^\bullet}^{\eta_\bullet}(t)\mathrm{d}\varkappa - \|B_+^-(\boldsymbol{\emptyset})\|_{\eta^\bullet}^{\eta_\bullet}(t) = \|B\|_{\eta^\bullet}^{\eta_\bullet}(t) - \|B_+^-(\boldsymbol{\emptyset})\|_{\eta^\bullet}^{\eta_\bullet}(t),$$
where $B_+^-(\varkappa, \boldsymbol{\vartheta}) = B(\boldsymbol{\varkappa})1_\emptyset(\vartheta_+^-)$, $\boldsymbol{\varkappa} = \begin{pmatrix} \vartheta_0^- & \varkappa \\ \vartheta_0^0 & \vartheta_+^0 \end{pmatrix}$. In the same way we obtain
$$\int_{X^t} \left(\|D_0^-(x)\|_{\xi^+}^{\xi_-}\right)^2 \mathrm{d}x \leq \int_{X^t} \left(\|\dot{B}_0^-(x)\|_{\eta^\bullet}^{\eta_\bullet}[t(x)]\right)^2 \mathrm{d}x \leq \eta^-(\|B\|_{\eta^\bullet}^{\eta_\bullet}(t))^2 \ ,$$
$$\int_{X^t} \left(\|D_+^0(x)\|_{\xi^+}^{\xi_-}\right)^2 \mathrm{d}x \leq \int_{X^t} \left(\|\dot{B}_+^0(x)\|_{\eta^\bullet}^{\eta_\bullet}[t(x)]\right)^2 \mathrm{d}x \leq \eta_+^{-1}(\|B\|_{\eta^\bullet}^{\eta_\bullet}(t))^2 \ ,$$
and
$$\operatorname{ess\,sup}_{x \in X^t} \|D_0^0(x)\|_{\xi^+}^{\xi_-} \leq \operatorname*{ess\,sup}_{x \in X^t} \|\dot{B}_0^0(x)\|_{\eta^\bullet}^{\eta_\bullet}[t(x)] \leq \sqrt{\eta^0/\eta_0} \ \|B\|_{\eta^\bullet}^{\eta_\bullet}(t) \ .$$
This proves the QS–integrability (1.3) of the derivatives $D_\nu^\mu(x)$ with respect to the (ξ^+, ξ_-) norms.

If $B(\boldsymbol{\vartheta})$ satisfies the recurrence (1.8), then $\dot{B}(\mathbf{x}_\nu^\mu) = L_\nu^\mu(x) \odot B$, and $D_\nu^\mu(x) = L_\nu^\mu(x) \odot U^{t(x)}$ due to the property $\imath_{[0,t)}^{\otimes}(L \odot B) = L \odot \imath_{[0,t)}^{\otimes}(B)$ for $L \odot B = (L \otimes \hat{1}) \cdot B$, following immediately from the definition (1.5). Hence $U^t = \imath_{[0,t)}^{\otimes}(B) = U^0 + \imath^t(\mathbf{D})$ with $U^0 = T^0 \otimes \hat{1}$ and $B(\boldsymbol{\vartheta})$, defined for the partitions $\varkappa = \sqcup \vartheta_\nu^\mu$ of the chains $\varkappa \in \mathcal{X}$ as $\mathbf{L}^\triangleleft(\boldsymbol{\vartheta}) \odot T^0$, where $\mathbf{L}^\triangleleft(\boldsymbol{\vartheta}) = L(\mathbf{x}_n) \cdots L(\mathbf{x}_1)$ for $\boldsymbol{\vartheta} = \sqcup_{i=1}^n \mathbf{x}_i$, satisfies the equation (1.10). ∎

Corollary 1. *Let $B(\boldsymbol{\vartheta}) = L(\boldsymbol{\vartheta}) \otimes \hat{1}$ be defined by the QS–integrable operator–valued function*
$$L \begin{pmatrix} \vartheta_0^- & \vartheta_+^- \\ \vartheta_0^0 & \vartheta_+^0 \end{pmatrix} : \mathcal{H} \otimes \mathcal{E}^\otimes(\vartheta_0^-) \otimes \mathcal{E}^\otimes(\vartheta_0^0) \to \mathcal{H} \otimes \mathcal{E}^\otimes(\vartheta_0^0) \otimes \mathcal{E}^\otimes(\vartheta_+^0)$$
with $\|L\|_{\eta^-,\eta^0}^{\eta_0,\eta_+} = \|B\|_{\eta^\cdot}^{\eta_\cdot} < \infty$ for $\eta^+, \eta_-^{-1} \geq 1$. Then the QS integral $\imath_{[0,t)}^{\otimes}(B) = U^t$ defines an adapted (ξ^+, ξ_-) continuous process U^t for $\xi^+ \geq \eta^- + \eta^0 + 1$, $\xi_-^{-1} \geq$

$\eta_+^{-1} + \eta_0^{-1} + 1$ in the sense $U^t(a^t \otimes c) = b^t \otimes c$ for all $a^t \in \mathcal{G}^t(\xi^+), c \in \mathcal{F}_{[t}$ with $b^t \in \mathcal{G}^t(\xi_-)$, where $\mathcal{G}^t(\xi) = \mathcal{H} \otimes \mathcal{F}^t(\xi)$,

$$\mathcal{F}^t(\xi) = \int_{\mathcal{X}^t}^{\oplus} \xi^{|\varkappa|} \mathcal{E}_\xi^{\otimes}(\varkappa) \mathrm{d}\varkappa, \quad \mathcal{X}^t = \{\varkappa \in \mathcal{X} | t(\varkappa) \subset [0,t)\},$$

It has the adapted QS derivatives $D_\nu^\mu(x) = A_\nu^\mu(x) \otimes \hat{1}_{[t(x)}$, where $\hat{1}_{[t}$ is the identity operator in $\mathcal{F}_{[t(x)}$ with $A_\nu^\mu(x)$, defined in $\mathcal{G}^{t(x)}$. If U^t is an adapted QS process with $\|U\|_{\xi,\infty}^{\xi_+,t} < \infty$, then $\mathrm{d}\imath^t(\mathbf{B} \odot U) = \mathrm{d}\imath^t(\mathbf{B})U^t$ in the sense

$$\imath^t(\mathbf{B} \odot U)a = \int_0^t \mathrm{d}\imath^s(\mathbf{B})U^s a,$$

where $\mathbf{B} \odot U = \mathbf{B} \cdot (U \otimes \mathbf{1})$ and the left hand side is defined as the limit of the Itô integral sums $\imath^t(\mathbf{B} \odot U) = \lim_{n\to\infty} \sum_{i=0}^n [\imath^{t_{i+1}}(\mathbf{B}) - \imath^{t_i}(\mathbf{B})] U^{t_i}$, $t_0 = 0, t_{n+1} = t$ in the uniform (ξ, ξ_-)–topology.

Indeed, the QS integral (1.5) for $B = L \otimes \hat{1}$ and $a = a^t \otimes c$ with $a^t \in \mathcal{G}^t$, $c \in \mathcal{F}_{[t}$ can be written as

$$[\imath_{[0,t)}^{\otimes}(B)a](\varkappa) = c(\varkappa_{[t}) \otimes \sum_{\sqcup\vartheta_\nu^0 = \varkappa^t} \int_{\mathcal{X}^t} \int_{\mathcal{X}^t} [L(\boldsymbol{\vartheta}) \otimes I(\vartheta_-^0)] a(\vartheta_0^- \sqcup \vartheta_0^0 \sqcup \vartheta_-^0) \mathrm{d}\vartheta_0^- \mathrm{d}\vartheta_+^- \ .$$

The norm $\|L \otimes \hat{1}\|_{\eta^\bullet}^{\eta_\bullet}$ for $\eta^+, \eta_-^{-1} \geq 1$ does not depend on η^+, η_-^{-1}, hence

$$\|\imath_{[0,t)}^{\otimes}(L \otimes \hat{1})\|_{\xi^+}^{\xi_-} \leq \|L\|_{\eta^-,\eta^0}^{\eta_0,\eta_+},$$

if $\xi^+ \geq \sum \eta^\mu$, $\xi_-^{-1} \geq \sum \eta_\nu^{-1}$ with $\eta^+ = 1 = \eta_-$.

The derivatives D_ν^μ are adapted as multiple QS integrals $\imath_{[o,t(x))}^{\otimes}(\dot{B}(\mathbf{x}_\nu^\mu))$ of $\dot{B}(\mathbf{x}) = \dot{L}(\mathbf{x}) \otimes \hat{1}$. If U^s , $s < t$ is a simple adapted function $U^s = \sum_{i=0}^n U_i 1_{[t_i,t_{i+1})}(s)$ with $t_0 = 0, t_{n+1} = t$, $1_{[t,t_+)}(s) = 1$ for $s \in [t, t_+)$, otherwise $1_{[t,t_+)}(s) = 0$, then

$$\imath^t(\mathbf{B} \odot U)a = \sum_{i=0}^n (\imath^{t_{i+1}} - \imath^{t_i})(\mathbf{B} \odot U_i)a = \sum_{i=0}^n [\imath^{t_{i+1}}(\mathbf{B}) - \imath^{t_i}(\mathbf{B})]b_i \ ,$$

where $b_i = U_i a$, if U is a constant adapted process on $[r, s)$:

$$[\imath_{[r,s)}^{\otimes}(BU)a](\varkappa) = \sum_{\vartheta_+^0 \sqcup \vartheta_0^0 \subseteq \varkappa_r^s} \int_{\mathcal{X}_r^s} \int_{\mathcal{X}_r^s} [B(\boldsymbol{\vartheta})\dot{b}(\vartheta_0^- \sqcup \vartheta_0^0)](\vartheta_-^0) \mathrm{d}\vartheta_0^- \mathrm{d}\vartheta_+^- \ .$$

3. A nonadapted QS calculus and Itô formula

Now we shall consider the operators $U = \epsilon(T)$ acting in $\mathcal{G} = \mathcal{H} \otimes \mathcal{F}$ as the multiple QS integrals (1.5) with $B = L \otimes \hat{1}$, and $t = \infty$ according to the formula

$$[\epsilon(T)a](\varkappa) = \sum_{\varkappa_0^0 \sqcup \varkappa_+^0 = \varkappa} \iint T(\boldsymbol{\varkappa}) a(\varkappa_0^0 \sqcup \varkappa_0^-) \mathrm{d}\varkappa_0^- \mathrm{d}\varkappa_+^- \qquad (2.1)$$

Here the sum is taken over all partitions of the chain $\varkappa \in \mathcal{X}$, and the operator-valued function $T(\boldsymbol{\varkappa})$ is in one to one correspondence

$$T\begin{pmatrix} \varkappa_0^- & \varkappa_+^- \\ \varkappa_0^0 & \varkappa_+^0 \end{pmatrix} = \sum_{\vartheta\subseteq\varkappa_0^0} L\begin{pmatrix} \varkappa_0^- & \varkappa_+^- \\ \vartheta & \varkappa_+^0 \end{pmatrix} \otimes I^{\otimes}(\varkappa_0^0\backslash\vartheta)$$
$$L\begin{pmatrix} \vartheta^- & \vartheta_+^- \\ \vartheta & \vartheta_+ \end{pmatrix} = \sum_{\varkappa\subseteq\vartheta} (-1)^{|\varkappa|} T\begin{pmatrix} \vartheta^- & \vartheta_+^- \\ \vartheta\backslash\varkappa & \vartheta_+ \end{pmatrix} \otimes I^{\otimes}(\varkappa)$$

with the operator-valued function $L(\boldsymbol{\vartheta})$, defining the integral representation $U = \Lambda_{[0,\infty)}(L\otimes\hat{1})$.

Using the arguments in section 1., one can prove, that the operator $\epsilon(T)$ is (ξ^+,ξ_-)-continuous, if T is $(\zeta^\bullet,\zeta_\bullet)$-bounded for $\zeta^\bullet = (\zeta^-,\zeta^0)$ and $\zeta_\bullet = (\zeta_0,\zeta_+)$, satisfying the inequalities $\zeta^- + \zeta^0 \le \xi^+$, $\zeta_0^{-1} + \zeta_+^{-1} \le \xi_-^{-1}$, because

$$\|\epsilon(T)\|_{\xi^+}^{\xi_-} \le \|T\|_{\zeta^\bullet}^{\zeta_\bullet} \equiv \int \left(\iint \frac{(\zeta_+)^{|\varkappa_+^0|}}{(\zeta^-)^{|\varkappa_0^-|}} \operatorname{ess\,sup}_{\varkappa_0^0\in\mathcal{X}} \frac{(\zeta_0)^{|\varkappa_0^0|}}{(\zeta^0)^{|\varkappa_0^0|}} \|T(\boldsymbol{\varkappa})\|^2 \mathrm{d}\varkappa_+^0 \mathrm{d}\varkappa_0^- \right)^{1/2} \mathrm{d}\varkappa_+^- .$$

In this case the formally conjugated operator

$$U^* = \epsilon(T^\star)\ ,\ T^\star(\boldsymbol{\varkappa}) = T(\boldsymbol{\varkappa}^\star)^*\ ,\ \begin{pmatrix} \varkappa_0^- & \varkappa_+^- \\ \varkappa_0^0 & \varkappa_+^0 \end{pmatrix}^\star = \begin{pmatrix} \varkappa_+^0 & \varkappa_+^- \\ \varkappa_0^0 & \varkappa_0^- \end{pmatrix} \tag{2.2}$$

exists as (ξ^+,ξ_-)-continuous operator $\mathcal{G}(\xi_+)\to\mathcal{G}(\xi^-)$ with $\|U^*\|_{\xi^+}^{\xi_-} = \|U\|_{1/\xi_-}^{1/\xi^+}$, if $\xi^+ \ge \zeta_0^{-1} + \zeta_+^{-1}$, $\xi_-^{-1} \ge \zeta^- + \zeta^0$.

As we shall prove now, the map ϵ is the representation in $\mathcal{G}$ of a unital $\star$-algebra of operator-valued functions $T(\boldsymbol{\varkappa})$, satisfying the relative boundedness condition

$$\|T\|(\boldsymbol{\zeta}) = \operatorname{ess\,sup}_{\boldsymbol{\varkappa}}\{\|T(\boldsymbol{\varkappa})\| / \prod_{\mu\le\nu} \zeta_\nu^\mu(\varkappa_\nu^\mu)\} < \infty\ , \tag{2.3}$$

where $\zeta(\varkappa) = \prod_{x\in\varkappa}\zeta(x)$, with respect to a triangular matrix-function $\boldsymbol{\zeta}(x) = [\zeta_\nu^\mu(x)]$, $\mu,\nu = -,0,+$ $\zeta_\nu^\mu = 0$ for $\mu>\nu$ under the order $- < 0 < +$, $\zeta_-^-(x) = 1 = \zeta_+^+(x)$ with positive L^p-integrable functions $(\zeta_\nu^\mu)_{\nu=0,+}^{\mu=-,0}$ for corresponding $p = 1,2,\infty$:

$$\|\zeta_+^-\|_1 \le \infty, \|\zeta_0^-\|_2 < \infty\ ,\ \|\zeta_+^0\|_2 < \infty\ ,\ \|\zeta_0^0\|_\infty < \infty\ ,$$

where $\|\zeta\|_p = (\int \zeta^p(x)\mathrm{d}x)^{1/p}$. In this case the operator $U = \epsilon(T)$ is ζ-bounded, as it follows from the next theorem, for $\zeta > \|\zeta_0^0\| = \operatorname{ess\,sup}_{x\in X}\zeta_0^0(x)$ in the sense of (ξ^+,ξ_-)-continuity of U for all $\xi_- > 0$, $\xi^+ \ge \zeta\cdot\xi_-$. This is due to the estimate

$$\|T\|_{\zeta^\bullet}^{\zeta_\bullet} \le \int \left(\iint \frac{(\zeta_+)^{|\varkappa_+^0|}}{(\zeta^-)^{|\varkappa_0^-|}} \operatorname{ess\,sup}_{\varkappa_0^0} \frac{(\zeta_0)^{|\varkappa_0^0|}}{(\zeta^0)^{|\varkappa_0^0|}} \left[\prod_{\mu<\nu}\zeta_\nu^\mu(\varkappa_\nu^\mu)\right]^2 \mathrm{d}\varkappa_+^0\mathrm{d}\varkappa_0^- \right)^{1/2} \mathrm{d}\varkappa_+^- \|T\|(\boldsymbol{\zeta})$$
$$= \int \prod_{x\in\varkappa}\zeta_+^-(x)\mathrm{d}\varkappa \left(\int \prod_{x\in\varkappa} \frac{\zeta_0^-(x)^2}{\zeta^-}\mathrm{d}\varkappa \int \prod_{x\in\varkappa} \frac{\zeta_+^0(x)^2}{\zeta_+^{-1}} \mathrm{d}\varkappa \operatorname{ess\,sup} \prod_{x\in\varkappa} \frac{\zeta_0^0(x)^2}{\zeta^0\zeta_0^{-1}} \right)^{1/2} \|T\|(\boldsymbol{\zeta})$$
$$\le \exp\{\int (\zeta_+^-(x) + (\zeta_0^-(x)^2 + \zeta_+^0(x)^2)/2\varepsilon)\mathrm{d}x\} \|T\|(\boldsymbol{\zeta})\ , \tag{2.4}$$

for $\zeta^-, \zeta_+^{-1} \geq \varepsilon > 0$, and $\zeta^0\zeta_0^{-1} \geq \|\zeta_0^0\|_\infty^2$, giving $\epsilon(T) = 0$, if $T(\boldsymbol{\varkappa}) = 0$ for almost all $\boldsymbol{\varkappa}$. Hence the operator (2.1) is defined even if $T(\boldsymbol{\varkappa})$ is described for almost all $\boldsymbol{\varkappa} = (\varkappa_\nu^\mu)$, in particular, only for the partitions $\varkappa = \sqcup\varkappa_\nu^\mu$ of the chains $\varkappa \in \mathcal{X}$.

Theorem 2. *If the operator-valued function*

$$T\begin{pmatrix} \varkappa_0^- & \varkappa_+^- \\ \varkappa_0^0 & \varkappa_+^0 \end{pmatrix} : \mathcal{H} \otimes \mathcal{E}^\otimes(\varkappa_0^-) \otimes \mathcal{E}^\otimes(\varkappa_0^0) \to \mathcal{H} \otimes \mathcal{E}^\otimes(\varkappa_0^0) \otimes \mathcal{E}^\otimes(\varkappa_+^0)$$

satisfies the condition (2.3), then the conjugated operators $U = \epsilon(T), U^* = \epsilon(T^\star)$ are ζ-bounded in $\mathcal{G}$ for any $\zeta > \zeta_0^0$, and the operator U^*U is defined in $\mathcal{G}$ as ζ^2-bounded operator

$$\epsilon(S \cdot T) = \epsilon(S)\epsilon(T) \ , \quad S = T^\star$$

by the following product formula

$$(S \cdot T)(\boldsymbol{\varkappa}) = \sum_{\vartheta_\nu^\mu \subseteq \varkappa_\nu^\mu}^{\mu<\nu} \sum_{\sigma_+^- \cap \rho_+^- = \vartheta_+^-}^{\sigma_+^- \cup \rho_+^- = \varkappa_+^-} S\begin{pmatrix} \vartheta_0^- \sqcup \vartheta_+^-, & \varkappa_+^- \setminus \sigma_+^- \\ \varkappa_0^0 \sqcup \vartheta_+^0, & \varkappa_+^0 \setminus \vartheta_+^0 \end{pmatrix} T\begin{pmatrix} \varkappa_0^- \setminus \vartheta_0^-, & \varkappa_+^- \setminus \rho_+^- \\ \varkappa_0^0 \sqcup \vartheta_0^-, & \vartheta_+^- \sqcup \vartheta_+^0 \end{pmatrix} . \tag{2.5}$$

This induces a unital $\star$-algebraic structure on the inductive space $\mathcal{U}$ of all relatively bounded functions T with

$$\|T^\star\|(\boldsymbol{\zeta}) = \|T\|(\boldsymbol{\zeta}^\star), \quad \|T^\star \cdot T\|(\boldsymbol{\xi}) \leq [\|T\|(\boldsymbol{\zeta})]^2,$$

if $\xi_\nu^\mu \geq (\boldsymbol{\zeta}^\star\boldsymbol{\zeta})_\nu^\mu$, where $\boldsymbol{\zeta}^\star(x) = \mathbf{g}\boldsymbol{\zeta}(x)^*\mathbf{g}$ and $(\boldsymbol{\zeta}^\star\boldsymbol{\zeta})(x) = \boldsymbol{\zeta}^\star(x)\boldsymbol{\zeta}(x)$ are defined by usual product of the matrices

$$\mathbf{g} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ \boldsymbol{\zeta}(x) = \begin{bmatrix} 1 & \zeta_0^- & \zeta_+^- \\ 0 & \zeta_0^0 & \zeta_+^0 \\ 0 & 0 & 1 \end{bmatrix}(x), \ \boldsymbol{\zeta}^\star(x) = \begin{bmatrix} 1 & \zeta_+^0 & \zeta_+^- \\ 0 & \zeta_0^0 & \zeta_0^- \\ 0 & 0 & 1 \end{bmatrix}(x) . \tag{2.6}$$

If the multiple QS integral $U^t = \Lambda_{[0,t)}(B)$ is defined by $B(\boldsymbol{\vartheta}) = \epsilon(L(\boldsymbol{\vartheta}))$ with

$$\left\| L\begin{pmatrix} \vartheta^- & \vartheta_+^- \\ \vartheta & \vartheta_+ \end{pmatrix} \right\|(\boldsymbol{\xi}) \leq c\lambda_0^0(\vartheta)\lambda_+^0(\vartheta_+)\lambda_0^-(\vartheta^-)\lambda_+^-(\vartheta_+), \ \lambda(\vartheta) = \prod_{x\in\vartheta}\lambda(x) \geq 0 \ ,$$

then $\Lambda_{[0,t)} \circ \epsilon = \epsilon \circ N_{[0,t)}$ i.e. $U^t = \epsilon(T^t)$, where

$$T^t(\boldsymbol{\varkappa}) = \sum_{\boldsymbol{\vartheta} \subseteq \boldsymbol{\varkappa}^t} L(\boldsymbol{\vartheta}, \boldsymbol{\varkappa}\setminus\boldsymbol{\vartheta}) \equiv N_{[0,t)}(L)(\boldsymbol{\varkappa}) \tag{2.7}$$

with $\|T^t\|(\boldsymbol{\zeta}) \leq c$, if $\zeta_\nu^\mu(x) \geq \xi_\nu^\mu(x) + \lambda_\nu^\mu(x)$ for $t(x) < t$, and $\zeta_\nu^\mu(x) \geq \xi_\nu^\mu(x)$ for $t(x) \geq t$. The QS derivatives $D_\nu^\mu(x) = \Lambda_{[0,t(x))}(\dot{B}(\mathbf{x}_\nu^\mu))$ for the process $U^t = \epsilon(T^t)$ have the natural difference form $\mathbf{D} = \mathbf{G} - \mathbf{U}$, described by the representations of

$$\dot{T}^t(\mathbf{x}, \boldsymbol{\varkappa}) = T^t(\boldsymbol{\varkappa} \sqcup \mathbf{x})$$

with $\mathbf{x} = \mathbf{x}_\nu^\mu, \mu < +, \nu > -$ at $t \searrow t(x)$

$$U_\nu^\mu(x) = \epsilon(\dot{T}^{t(x)}(\mathbf{x}_\nu^\mu)), \quad G_\nu^\mu(x) = \epsilon(\dot{T}^{t(x)]}(\mathbf{x}_\nu^\mu)) \ , \tag{2.8}$$

where $T^{s]}(\boldsymbol{\varkappa}) = N_{[0,t)}(L)(\boldsymbol{\varkappa})$ for any $t > s$, $t \leq t(x)$ $\forall x \in \varkappa_s = \sqcup\varkappa_\nu^\mu \cap t^{-1}(s, \infty)$, and $\mathbf{x}_\nu^\mu$ denotes an elementary table $\boldsymbol{\vartheta} = (\vartheta_\lambda^\kappa)$ with $\vartheta_\lambda^\kappa = \emptyset$ except $\vartheta_\nu^\mu = x$. The QS differential $\mathrm{d}U^* = \mathrm{d}\Lambda(\mathbf{D}^\star)$ is defined by the derivative $\mathbf{D}^\star = \mathbf{G}^\star - \mathbf{U}^\star$, and

$$\mathrm{d}(U^*U) = \mathrm{d}\Lambda(\mathbf{U}^\star\mathbf{D} + \mathbf{D}^\star\mathbf{U} + \mathbf{D}^\star\mathbf{D}) = \mathrm{d}\Lambda(\mathbf{G}^\star\mathbf{G} - \mathbf{U}^\star\mathbf{U}) \ , \tag{2.9}$$

where the QS derivative $\mathbf{G}^\star\mathbf{G}-\mathbf{U}^\star\mathbf{U}$ *of the QS process* $(U^*U)^t = U^{t*}U^t$ *is described in terms of the usual products* $(\mathbf{U}^\star\mathbf{U})(x) = \mathbf{U}^\star(x)\mathbf{U}(x)$ *and the pseudo Hermitian conjugation* $\mathbf{U}^\star(x) = (\hat{I}\otimes\mathbf{g}(x))\mathbf{U}(x)^*(\hat{I}\otimes\mathbf{g}(x))$ *of the triangular matrices*

$$\mathbf{U} = \begin{bmatrix} U & U_0^- & U_+^- \\ 0 & U_0^0 & U_+^0 \\ 0 & 0 & U \end{bmatrix}, \ \mathbf{D} = \begin{bmatrix} 0 & D_0^- & D_+^- \\ 0 & D_0^0 & D_+^0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} U, & G_0^-, & G_+^- \\ 0 & G_0^0 & G_+^0 \\ 0 & 0 & U \end{bmatrix},$$

with $U(x) = U^{t(x)}, \mathbf{g}(x) = [g_\nu^\mu(x)], g_-^- = 1 = g_+^+, g_0^0(x) = I(x)$*, otherwise* $g_\nu^\mu = 0$.

Proof. Let us firstly obtain an estimate for the representation $U = \epsilon(T)$ of a relatively bounded operator-valued function T in the sense (2.3). Due to the inequalities (2.4) one obtains

$$\|U\|_{\xi^+}^{\xi} \leq \exp\{\|\zeta_+^-\|_1 + (\|\zeta_0^-\|_2^2 + \|\zeta_+^0\|_2^2)/2\varepsilon\}\|T\|(\boldsymbol{\zeta}) \ ,$$

if

$$\xi^+ \geq \varepsilon + \zeta^0, \quad \xi_-^{-1} \geq \zeta_0^{-1} + \varepsilon \ , \ \zeta^0\zeta_0^{-1} \geq \|\zeta_0^0\|_\infty^2 \ .$$

Hence for any $\xi^+\xi_-^{-1} > \|\zeta_0^0\|_\infty^2$ there exists an $\varepsilon > 0$ such that this inequality holds, namely,

$$\varepsilon \leq \left(\xi^+ + \xi_-^{-1} - \sqrt{(\xi^+ - \xi_-^{-1})^2 + 4\|\zeta_0^0\|_\infty^2}\right) \qquad /2 = \varepsilon(\xi^+, \xi_-) \ ,$$

where the upper bound $\varepsilon(\xi^+, \xi_-)$ corresponds to the solution $\varepsilon > 0$ of the equation $\zeta^0\zeta_0^{-1} = \|\zeta_0^0\|_\infty^2$ with $\zeta^0 = \xi^+ - \varepsilon > 0$, $\zeta_0^{-1} = \xi_-^{-1} - \varepsilon > 0$. Hence the operator U is ζ-bounded for any $\zeta > \zeta_0^0$ and also U^* is ζ-bounded due to $(\boldsymbol{\zeta}^\star)_0^0 = \zeta_0^0 = (\boldsymbol{\zeta})_0^0$.

Now we show that the product formula (2.3) is valid for $T(\varkappa) = X \otimes \mathbf{f}^\otimes(\boldsymbol{\varkappa})$, where $X \in \mathcal{B}(\mathcal{H})$, and

$$\mathbf{f}^\otimes(\boldsymbol{\varkappa}) = \otimes_{\mu\leq\nu} f_\nu^\mu(\varkappa_\nu^\mu)$$

with $f_\nu^\mu(\varkappa) = \otimes_{x\in\varkappa} f_\nu^\mu(x)$ defined by the operator-valued elements $(f_\nu^\mu)_{\nu=0,+}^{\mu=-,0}$ of the matrix-function

$$\mathbf{f}(x) = \begin{bmatrix} 1 & f_0^- & f_+^- \\ 0 & f_0^0 & f_+^0 \\ 0 & 0 & 1 \end{bmatrix}(x), \qquad \begin{matrix} f_0^0(x) : \mathcal{E}(x) \to \mathcal{E}(x) \ , \ f_+^-(x) : \mathbb{C} \to \mathbb{C} \\ f_+^0(x) : \mathbb{C} \to \mathcal{E}(x) \ , \ f_0^-(x) : \mathcal{E}(x) \to \mathbb{C} \end{matrix}$$

with

$$\|f_+^-\|_1 < \infty, \quad \|f_0^-\|_2 < \infty, \quad \|f_+^0\|_2 < \infty, \quad \|f_0^0\|_\infty < \infty \ .$$

Let us find the action (2.1) of the operator $U = \epsilon(X \otimes \mathbf{f}^\otimes)$ on the product vector $a = h \otimes k^\otimes \ , \ h \in \mathcal{H} \ , \ k \in \mathcal{K}$, where $k^\otimes(\varkappa) = \otimes_{x\in\varkappa} k(x)$:

$$[Ua](\varkappa) = Xh \otimes \sum_{\varkappa_0^0 \sqcup \varkappa_+^0 = \varkappa} \iint f_+^-(\varkappa_+^-) f_0^-(\varkappa_0^-) k^\otimes(\varkappa_0^-) f_+^0(\varkappa_+^0) \otimes f_0^0(\varkappa_0^0) k^\otimes(\varkappa_0^0) \mathrm{d}\varkappa_+^- \mathrm{d}\varkappa_0^-$$

$$= Xh \otimes \sum_{\varkappa_0^0 \sqcup \varkappa_+^0 = \varkappa} \otimes_{x\in\varkappa_+^0} f_+^0(x) \otimes_{x\in\varkappa_0^0} f_0^0(x) k(x) \int \prod_{x\in\varkappa} f_+^-(x) \mathrm{d}\varkappa \int \prod_{x\in\varkappa} f_0^-(x) k(x) \mathrm{d}\varkappa$$

$$= Xh \otimes (f_+^0 + f_0^0 k)^\otimes(\varkappa) \exp\{\int (f_+^-(x) + f_0^-(x) k(x)) \mathrm{d}x\}$$

In the same way, acting on the product vector $Xh \otimes (f_+^0 + f_0^0 k)^\otimes$ by $U^* = \epsilon(X^* \otimes \mathbf{f}^{\star\otimes})$ with

$$\mathbf{f}^\star(x)_0^0 = f_0^0(x)^*, \quad \mathbf{f}^\star(x)_+^- = f_+^-(x)^*, \quad \mathbf{f}^\star(x)_+^0 = f_0^-(x)^*, \quad \mathbf{f}^\star(x)_0^- = f_+^0(x)^* \ ,$$

one obtains $[U^*Ua](\varkappa) = X^*Xh \otimes (f_0^{-*} + f_0^{0*}(f_+^0 + f_0^0 k))^{\otimes}(\varkappa)$.

$$\exp\{\int [f_+^-(x)^* + f_+^{0*}(x)(f_+^0(x) + f_0^0(x)k(x)) + f_+^-(x) + f_0^-(x)k(x)]\mathrm{d}x\} =$$

$$= X^*Xh \otimes ((\mathbf{f}^\star\mathbf{f})_+^0 + (\mathbf{f}^\star\mathbf{f})_0^0 k)^{\otimes}(\varkappa) \exp\{\int ((\mathbf{f}^\star\mathbf{f})_+^-(x) + (\mathbf{f}^\star\mathbf{f})_0^- k(x))\mathrm{d}x\} \ ,$$

where the operator-valued functions

$$(\mathbf{f}^\star\mathbf{f})_0^0(x) = f_0^0(x)^* f_0^0(x), (\mathbf{f}^\star \ \mathbf{f})_+^-(x) = f_+^-(x)^* + f_+^0(x)^* f_+^0(x) + f_+^-(x)$$

$$(\mathbf{f}^\star\mathbf{f})_+^0(x) = f_0^-(x)^* + f_0^0(x)^* f_+^0(x), (\mathbf{f}^\star\mathbf{f})_0^-(x) = f_+^0(x)^* f_0^0(x) + f_0^-(x)$$

are defined as matrix elements of the product $(\mathbf{f}^\star\mathbf{f})(x) = \mathbf{f}(x)^\star\mathbf{f}(x)$ of triangular matrices $\mathbf{f}^\star$ and $\mathbf{f}$. Hence on the linear span of the product vectors $a = h \otimes k^{\otimes}$ we have for $T = X \otimes \mathbf{f}^\star$ the $\star$-multiplicative property

$$\epsilon(T)^*\epsilon(T) = \epsilon(X^*X \otimes (\mathbf{f}^\star\mathbf{f})^{\otimes}) = \epsilon(T^\star \cdot T) \ ,$$

where the product $(T^\star \cdot T)(\boldsymbol{\varkappa})$ is defined as (2.5) due to $(\mathbf{f}^\star\mathbf{f})^{\otimes} = \mathbf{f}^{\otimes\star} \cdot \mathbf{f}^{\otimes}$:

$$(\mathbf{f}^\star\mathbf{f})^{\otimes}(\boldsymbol{\varkappa}) = \otimes_{x\in\varkappa_0^0}(f_0^0(x)^* f_0^0(x)) \otimes_{x\in\varkappa_+^0} (f_0^-(x)^* + f_0^0(x)^* f_+^0(x))\otimes$$

$$\otimes_{x\in\varkappa_0^-}(f_+^0(x)^* f_0^0(x) + f_0^-(x)) \otimes_{x\in\varkappa_+^-} (f_+^-(x)^* + f_+^0(x)^* f_+^0(x) + f_+^-(x)) =$$

$$= \sum_{\vartheta_\nu^\mu \subseteq \varkappa_\nu^\mu}^{\mu<\nu} f_0^0(\varkappa_0^0)^* f_0^0(\varkappa_0^0) \otimes f_0^-(\varkappa_+^0 \backslash \vartheta_+^0)^* \otimes f_0^0(\vartheta_+^0)^* f_+^0(\vartheta_+^0)\otimes$$

$$f_+^0(\vartheta_0^-)^* f_0^0(\vartheta_0^-)) \otimes f_0^-(\varkappa_0^- \backslash \vartheta_0^-) \sum_{\sigma_+^- \cap \rho_+^- = \vartheta_+^-}^{\sigma_+^- \cup \rho_+^- = \varkappa_+^-} f_+^-(\varkappa_+^- \backslash \sigma_+^-)^* f_+^0(\vartheta_+^-)^* f_+^0(\vartheta_+^-) f_+^-(\varkappa_+^- \backslash \rho_+^-)$$

$$= \sum_{\vartheta_\nu^\mu \subseteq \varkappa_\nu^\mu}^{\mu<\nu} \sum_{\sigma_+^- \cap \rho_+^- = \vartheta_+^-}^{\sigma_+^- \cup \rho_+^- = \varkappa_+^-} \mathbf{f}^{\otimes} \begin{pmatrix} \varkappa_+^0 \backslash \vartheta_+^0 & \varkappa_+^- \backslash \sigma_+^- \\ \varkappa_0^0 \sqcup \vartheta_+^0 & \vartheta_0^- \sqcup \vartheta_+^- \end{pmatrix}^* \mathbf{f}^{\otimes} \begin{pmatrix} \varkappa_0^- \backslash \vartheta_0^- & \varkappa_+^- \backslash \rho_+^- \\ \varkappa_0^0 \sqcup \vartheta_0^- & \vartheta_+^- \sqcup \vartheta_+^0 \end{pmatrix} \ .$$

As the operator-valued functions $X \otimes \mathbf{f}^{\otimes}(\boldsymbol{\varkappa})$ are relatively bounded $\|X \otimes \mathbf{f}^{\otimes}(\boldsymbol{\varkappa})\|(\boldsymbol{\zeta}) = \|X\|$ with respect to $\zeta_\nu^\mu(x) = \|f_\nu^\mu(x)\|$ and their linear span is dense in inductive space $\mathcal{U}$, the product formula can be obtained as a limit for any $T \in \mathcal{U}$, and

$$\|(T^\star \cdot T)(\boldsymbol{\varkappa})\| \le \sum \|T^\star \begin{pmatrix} \vartheta_0^- \sqcup \vartheta_+^- & \varkappa_+^- \backslash \sigma_+^- \\ \varkappa_0^0 \sqcup \vartheta_+^0 & \varkappa_+^0 \backslash \vartheta_+^0 \end{pmatrix} \| \, \|T \begin{pmatrix} \varkappa_0^- \backslash \vartheta_0^- & \varkappa_+^- \backslash \rho_+^- \\ \varkappa_0^0 \backslash \vartheta_0^- & \vartheta_+^- \sqcup \vartheta_+^0 \end{pmatrix} \| \le$$

$$\|T\|^2(\boldsymbol{\zeta}) \sum \boldsymbol{\zeta}^{\otimes} \begin{pmatrix} \varkappa_+^0 \backslash \vartheta_+^0, & \varkappa_+^- \backslash \sigma_+^- \\ \varkappa_0^0 \sqcup \vartheta_+^0, & \vartheta_0^- \sqcup \vartheta_+^- \end{pmatrix} \boldsymbol{\zeta}^{\otimes} \begin{pmatrix} \varkappa_0^- \backslash \vartheta_0^- & \varkappa_+^- \backslash \rho_+^- \\ \varkappa_0^0 \backslash \vartheta_0^- & \vartheta_+^- \sqcup \vartheta_+^0 \end{pmatrix} = [\|T\|(\boldsymbol{\zeta})]^2 (\boldsymbol{\zeta}^2)^{\otimes}(\boldsymbol{\varkappa})$$

this means $\|T^\star \cdot T\|(\boldsymbol{\zeta}^\star\boldsymbol{\zeta}) \le [\|T\|(\boldsymbol{\zeta})]^2$. Due to the proven continuity of the linear map ϵ on $\mathcal{U}$ into the $*$-algebra of relatively bounded operators on the projective limit $\cap_{\xi>0}\mathcal{G}(\xi)$, the $\star$-multiplicative property of ϵ can be extended on the whole $\star$-algebra $\mathcal{U}$ with the unity $I(\boldsymbol{\varkappa}) = I \otimes \mathbf{1}^{\otimes}(\boldsymbol{\varkappa})$, $\mathbf{1}(x)$ is the identity matrix, having the representation $\epsilon(I) = \hat{I}$.

Now let us find the representation U^t of the multiple quantum integral (2.7), having the values $\epsilon \circ N_{[0,t)}(L)$ in $\mathcal{U}$ for relatively bounded operator-valued functions $L(\boldsymbol{\vartheta}, \boldsymbol{\varkappa})$ due to

$$\|T^t(\boldsymbol{\varkappa})\| \leq c \sum_{\vartheta_0^0 \subseteq \varkappa_0^0}^{t(\vartheta_0^0)<t} \sum_{\vartheta_+^0 \subseteq \varkappa_+^0}^{t(\vartheta_+^0)<t} \sum_{\vartheta_0^- \subseteq \varkappa_0^-}^{t(\vartheta_0^-)<t} \sum_{\vartheta_+^- \subseteq \varkappa_+^-}^{t(\vartheta_+^-)<t} \|L(\boldsymbol{\vartheta}, \boldsymbol{\varkappa} \backslash \boldsymbol{\vartheta})\|$$

$$\leq c \prod_{\nu=0,+}^{\mu=-,0} \sum_{\vartheta_\nu^\mu \leq \varkappa_\nu^\mu}^{t(\vartheta_\nu^\mu)<t} \lambda_\nu^\mu(\vartheta_\nu^\mu) \xi_\nu^\mu(\varkappa_\nu^\mu \backslash \vartheta_\nu^\mu) = c \prod_{\nu=0,+}^{\mu=-,0} \zeta_\nu^\mu(\varkappa_\nu^\mu) \ ,$$

where

$$\zeta(\varkappa) = \prod_{x \in \varkappa}^{t(x)<t} [\lambda(x) + \xi(x)] \prod_{x \in \varkappa}^{t(x) \geq t} \xi(x)$$

for

$$\lambda(\vartheta) = \prod_{x \in \vartheta} \lambda(x), \quad \xi(\varkappa) = \prod_{x \in \varkappa} \xi(x) \ .$$

From the definitions (2.1) of $U^t = \epsilon(T^t)$ we obtain the QS integral (1.5):

$$[U^t a](\varkappa) = \sum_{\varkappa_0^0 \sqcup \varkappa_+^0 = \varkappa} \iint \sum_{\boldsymbol{\vartheta} \subseteq \boldsymbol{\varkappa}^t} L(\boldsymbol{\vartheta}, \boldsymbol{\varkappa} \backslash \boldsymbol{\vartheta}) a(\varkappa_0^0 \sqcup \varkappa_0^-) \mathrm{d}\varkappa_0^- \mathrm{d}\varkappa_+^-$$

$$= \sum_{\vartheta \sqcup \vartheta_+ \subseteq \varkappa^t} \int_{\mathcal{X}^t} \int_{\mathcal{X}^t} \sum_{\varkappa_0^0 \sqcup \varkappa_+^0 = \vartheta_-} \iint L(\boldsymbol{\vartheta}, \boldsymbol{\varkappa}) \dot{a}(\vartheta \sqcup \vartheta^-, \varkappa_0^0 \sqcup \varkappa_0^-) \mathrm{d}\varkappa_0^- \mathrm{d}\varkappa_+^- \mathrm{d}\vartheta^- \mathrm{d}\vartheta_+^- \ ,$$

where $\vartheta_- = \varkappa \backslash (\vartheta \sqcup \vartheta_+)$, $\dot{a}(\vartheta, \varkappa_0^0) = a(\vartheta \sqcup \varkappa_0^0)$. Hence $\epsilon(T^t) = \Lambda_{[0,t)}(B)$ with

$$[B(\boldsymbol{\vartheta}) \dot{a}(\vartheta \sqcup \vartheta^-)](\varkappa) = \sum_{\varkappa_0^0 \sqcup \varkappa_+^0 = \varkappa} \iint L(\boldsymbol{\vartheta}, \boldsymbol{\varkappa}) \dot{a}(\vartheta \sqcup \vartheta^-, \varkappa_0^0 \sqcup \varkappa_0^-) \mathrm{d}\varkappa_0^- \mathrm{d}\varkappa_+^- \ ,$$

that is $B(\boldsymbol{\vartheta}) = \epsilon(L(\boldsymbol{\vartheta}))$. In particular, if $U^t = U^0 + \Lambda^t(\mathbf{D})$ with $U^0 = \epsilon(T^0)$ and $\mathbf{D}(x) = \epsilon(\mathbf{C}(x))$, then $U^t = \epsilon(T^0 + N^t(\mathbf{C}))$, i.e. $\epsilon \circ N^t = \Lambda^t \circ \epsilon$, where

$$N^t(\mathbf{C})(\boldsymbol{\varkappa}) = \sum_{x \in \boldsymbol{\varkappa}^t} C(\mathbf{x}, \boldsymbol{\varkappa} \backslash \mathbf{x}), \quad C(\mathbf{x}_\nu^\mu, \boldsymbol{\varkappa}) = C_\nu^\mu(x, \boldsymbol{\varkappa}) \ .$$

In the case $U^t = \Lambda_{[0,t)}(B)$ with $B = \epsilon(L)$ the QS derivatives

$$D_\nu^\mu(x) = \Lambda_{[0,t(x))}(\dot{B}(\mathbf{x}_\nu^\mu)) = \epsilon(C_\nu^\mu(x))$$

are defined by

$$C_\nu^\mu(x, \boldsymbol{\varkappa}) = N_{[0,t(x))}(\dot{L}(\mathbf{x}_\nu^\mu) = \dot{T}^{t(x)]}(\mathbf{x}_\nu^\mu, \boldsymbol{\varkappa}) - \dot{T}^{t(x)}(\mathbf{x}_\nu^\mu, \boldsymbol{\varkappa}) \ ,$$

where

$$N_{[0,t)}(\dot{L}(\mathbf{x})) = \sum_{\boldsymbol{\vartheta} \subseteq \boldsymbol{\varkappa}^t} L(\boldsymbol{\vartheta} \sqcup \mathbf{x}, \boldsymbol{\varkappa} \backslash \boldsymbol{\vartheta}),$$

$\mathbf{x}$ is one of the four elementary tables $\mathbf{x}_\nu^\mu$ and

$$\dot{T}^{t(x)}(\mathbf{x}, \boldsymbol{\varkappa}) = \sum_{\boldsymbol{\vartheta} \subseteq \boldsymbol{\varkappa}^{t(x)}} L(\boldsymbol{\vartheta}, \boldsymbol{\varkappa} \sqcup \mathbf{x}) \backslash \boldsymbol{\vartheta}) = T^{t(x)} \quad (\boldsymbol{\varkappa} \sqcup \mathbf{x}) \ ,$$

$$\dot{T}^{t(x)]}(\mathbf{x}, \boldsymbol{\varkappa}) = \sum_{\boldsymbol{\vartheta} \subseteq \boldsymbol{\varkappa}^{t(x)} \sqcup \mathbf{x}} L(\boldsymbol{\vartheta}, \boldsymbol{\varkappa} \sqcup \mathbf{x}) \backslash \boldsymbol{\vartheta}) = T^{t(x)} \quad (\boldsymbol{\varkappa} \sqcup \mathbf{x})$$

$$+\sum_{\boldsymbol{\vartheta}\subseteq\varkappa^{t(x)}} L(\boldsymbol{\vartheta}\sqcup \mathbf{x},\boldsymbol{\varkappa}\backslash\boldsymbol{\vartheta}) = \dot{T}^{t(x)}(\mathbf{x},\boldsymbol{\varkappa}) + N_{[0,t(x))}(\dot{L}(\mathbf{x}))(\boldsymbol{\varkappa})$$

due to

$$T^{s]}(\boldsymbol{\varkappa}) = \sum_{\boldsymbol{\vartheta}\subseteq\varkappa^{s]}} L(\boldsymbol{\vartheta},\boldsymbol{\varkappa}\backslash\boldsymbol{\vartheta}) = T^{s_+}(\boldsymbol{\varkappa})$$

where

$$\varkappa^{s]} = \{x\in\varkappa | t(x)\le s\}\ , \quad s_+ = \min\{t(x) > s | x\in\varkappa\}\ .$$

Hence the derivatives $D^\mu_\nu(x), x\in X^t$, defining $U^t - U^0 = \Lambda_{[0,t)}(\mathbf{D})$ are represented as the differences

$$D^\mu_\nu(x) = \epsilon[\dot{T}^{t(x)]}(\mathbf{x}^\mu_\nu)] - \epsilon[\dot{T}^{t(x)}(\mathbf{x}^\mu_\nu)]$$

of the operators (2.8), where $\dot{T}^{s]}(\mathbf{x},\boldsymbol{\varkappa}) = T^t(\boldsymbol{\varkappa}\sqcup\mathbf{x}) = \dot{T}^t(\mathbf{x},\boldsymbol{\varkappa})$ for any $t : s < t \le s_+ = \min\{t(x) > s | x\in\boldsymbol{\varkappa}\}$.

Let us consider $\dot{T}^t(\mathbf{x})$ as elements $T^\mu_\nu(x) = \dot{T}(\mathbf{x}^\mu_\nu)$ of the triangular operator-valued matrix-function $\mathbf{T}^t(x)$ with $T^\mu_\nu = 0$ for $\mu > \nu$, and $T^-_-(x) = T^t = T^+_+(x)$ independent of $x\in X$, defining the triangular matrices $\mathbf{U} = [U^\mu_\nu]$ and $\mathbf{G} = [G^\mu_\nu]$ as $\mathbf{U}(x) = \epsilon(\mathbf{T}^{t(x)}(x))$ and $\mathbf{G}(x) = \epsilon(\mathbf{T}^{t(x)]}(x))$, where $\mathbf{T}^{s]} = \mathbf{T}^t$ for a $t\in(s,s_+]$. This helps to generalize the QS Itô formula [1] for nonadapted processes as

$$U^{t*}U^t - U^{0*}U^0 = \Lambda^t(\mathbf{U}^\star\mathbf{D} + \mathbf{D}^\star\mathbf{U} + \mathbf{D}^\star\mathbf{D}),$$

because

$$U^{t*}U^t = \epsilon(T^{t\star}\cdot T^t), (T^\star\cdot T)(\boldsymbol{\varkappa}\sqcup\mathbf{x}^\mu_\nu) = (\mathbf{T}^\star(x)\cdot\mathbf{T}(x))^\mu_\nu(\boldsymbol{\varkappa})\ ,$$

as it follows directly from the formula (2.5), in terms of the usual product of triangular matrices $T^\star$ and T, defined by the multiplication $\cdot$ of the matrix elements $T^\mu_\nu(x)$ and $\star$-multiplicative property

$$\epsilon(\mathbf{T}^\star(x)\cdot\mathbf{T}(x)) \ = \ \epsilon(\mathbf{T}(x))^\star\epsilon(\mathbf{T}(x))\ .$$

Applying this for $t = t(x)$ and $t = t_+(x) = \min\{t\in t(\varkappa) | t > t(x)\}$ to the representation

$$\epsilon[(\mathbf{T}^{t(x)]\star}\mathbf{T}^{t(x)]})(x) - (\mathbf{T}^{t(x)\star}\mathbf{T}^{t(x)})(x)]$$

of the QS derivative of the process $U^{t*}U^t$, we finally obtain the formula

$$\mathrm{d}(U^{t*}U^t) = \mathrm{d}\Lambda^t[\epsilon(\mathbf{T}^{t]})^\star\epsilon(\mathbf{T}^{t]}) - \epsilon(\mathbf{T}^t)^\star\epsilon(\mathbf{T}^t)]\ ,$$

giving the multiplication table (2.9) in terms of the triangular matrices (2.8) with $G^-_-(x) = U^{t(x)} = G^+_+(x)$,$U^-_-(x) = U^{t(x)} = U^+_+(x)$ and $D^\mu_\nu(x) = G^\mu_\nu(x) - U^\mu_\nu(x)$. ∎

Corollary 2. *The QS process $U^t = \epsilon(T^t)$ is adapted, iff $T^t(\boldsymbol{\varkappa}) = T(\boldsymbol{\varkappa}^t)\otimes 1(\boldsymbol{\varkappa}_{[t})$ for almost all $\boldsymbol{\varkappa} = (\varkappa^\mu_\nu)$, where $\boldsymbol{\varkappa}^t = \boldsymbol{\varkappa}\cap X^t, \boldsymbol{\varkappa}_{[t} = \boldsymbol{\varkappa}\cap X_{[t}$, and $1(\boldsymbol{\varkappa}) = I(\varkappa^0_0)$ for $\varkappa^\mu_\nu = \emptyset, \mu\ne\nu$, otherwise $1(\boldsymbol{\varkappa}) = 0$. The QS Itô formula for adapted processes U^t can be written in the form*

$$\mathrm{d}(U^*U) = \mathrm{d}\Lambda(\mathbf{G}^\star\mathbf{G} - \mathbf{U}^*\mathbf{U}\otimes\mathbf{1}) = U^*\mathrm{d}U + \mathrm{d}U^*U + \mathrm{d}U^*\mathrm{d}U\ , \tag{2.10}$$

where $\mathrm{d}U^*\mathrm{d}U = \mathrm{d}\Lambda(\mathbf{D}^\star\mathbf{D})$ is defined by the usual product of the triangular matrices $\mathbf{D} = [D^\mu_\nu]$, $\mathbf{D}^\star = (\hat{I}\otimes\mathbf{g})\mathbf{D}^*(\hat{I}\otimes\mathbf{g})$ with $D^\mu_\nu = 0$, if $\mu = +$ or $\nu = -$.

Indeed, if $T^t(\boldsymbol{\varkappa}) = T(\boldsymbol{\varkappa}^t) \otimes 1(\boldsymbol{\varkappa}_{[t})$, then

$$[U^t(a^t \otimes c)](\varkappa) = \sum_{\varkappa_0^0 \sqcup \varkappa_+^0 = \varkappa^t} \iint T(\boldsymbol{\varkappa}^t) a(\varkappa_0^0 \sqcup \varkappa_0^-) \otimes c(\varkappa_{[t}) \mathrm{d}\varkappa_0^- \mathrm{d}\varkappa_+^- \ ,$$

for any $a^t \in \mathcal{G}^t, c \in \mathcal{F}_{[t}$, where the integral should be taken over $\varkappa_0^-, \varkappa_+^- \in \mathcal{X}^t$, otherwise $T^t(\boldsymbol{\varkappa}) = 0$. Hence $U^t(a^t \otimes c) = b^t \otimes c$ for a $b^t \in \mathcal{G}^t$. In this case

$$\dot{T}^{t(x)}(\mathbf{x}_\nu^\mu, \boldsymbol{\varkappa}) = T^{t(x)}(\boldsymbol{\varkappa} \sqcup \mathbf{x}_\nu^\mu) = T(\boldsymbol{\varkappa}^{t(x)}) \otimes 1_\nu^\mu(x) \otimes 1(\boldsymbol{\varkappa}_{t(x)}) \ ,$$

where $1_\nu^\mu(x) = 0$, if $\mu \neq \nu, 1_-^- = 1 = 1_+^+, 1_0^0(x) = I(x)$. This gives $U_\nu^\mu(x) = \epsilon(\dot{T}^{t(x)}(\mathbf{x}_\nu^\mu)) = U^t \otimes 1_\nu^\mu(x)$, and

$$\mathrm{d}\Lambda(G^\star G - U^* U \otimes \mathbf{1}) = \mathrm{d}\Lambda((U^* \otimes \mathbf{1})\mathbf{D} + \mathbf{D}^\star(U \otimes \mathbf{1}) + \mathbf{D}^\star \mathbf{D}) =$$

$$U^* \mathrm{d}\Lambda(\mathbf{D}) + \mathrm{d}\Lambda(\mathbf{D}^\star) U + \mathrm{d}\Lambda(\mathbf{D}^\star \mathbf{D}) = U^* \mathrm{d}U + \mathrm{d}U^* U + \mathrm{d}U^* \mathrm{d}U$$

as it follows from corollary 1 for the adaptive U^t.

4. A nonadapted QS evolution and chronological products

The proved $\star$-homomorphism and continuity properties of the representation ϵ of the unital inductive $\star$-algebra $\mathcal{U}$ of all operator-valued functions $T(\boldsymbol{\varkappa})$ of $\varkappa_\nu^\mu \in \mathcal{X}$, $(\mu, \nu) \in \{-, 0\} \times \{0, +\}$, relatively bounded with respect to some $\boldsymbol{\zeta} = [\zeta_\nu^\mu(x)]$, into the $\star$-algebra $\mathcal{B}$ of all relatively bounded operators on the projective limit $\mathcal{G}^+ = \bigcap_{\xi > 0} \mathcal{G}(\xi)$ enables us to construct a QS functional calculus.

Namely, if $T = f(Q_1, \dots, Q_m)$ is an analytical function of $Q_i \in \mathcal{U}$ as a limit in $\mathcal{U}$ of polynomials T_n with some ordering of noncommuting $Q_1, \dots, Q_m$ in the sense $\|T_n - T\|(\boldsymbol{\zeta}) \to 0$ for a $\boldsymbol{\zeta}$, then $U = \epsilon(T)$ is the ordered function $f(X_1, \dots, X_m)$ of $X_i = \epsilon(Q_i)$ as a limit on $\mathcal{G}^+$ of the corresponding polynomials $U_n = \epsilon(T_n)$, that is $\|U_n - U\|_{\xi^+}^{\xi_-} \to 0$ for any $\xi_- > 0$ and $\xi^+ > \xi_- \|\zeta_0^0\|_\infty^2$. The function $U^* = f^*(X_1^*, \dots, X_n^*)$ with the transposed ordering as $U^* = \epsilon(T^\star)$ for $T^\star = f^*(Q_1^\star, \dots, Q_n^\star)$ is also defined as (ξ^+, ξ_-)-limit due to $\|T_n^\star - T^\star\|(\boldsymbol{\zeta}^\star) \to 0$ and $(\boldsymbol{\zeta}^\star)_0^0 = \zeta_0^0$.

The differential form of this calculus is given by the noncommutative and nonadaptive generalization of the QS Itô formula

$$\mathrm{d}X = \mathrm{d}\Lambda(\mathbf{A}) \Rightarrow \mathrm{d}f(X) = \mathrm{d}\Lambda(f(\mathbf{X} + \mathbf{A}) - f(\mathbf{X})) \tag{3.1}$$

defined for any analytical function $U^t = f(X^t)$ of $\epsilon(Q^t)$ as QS differential of $\epsilon(T^t)$ for $T^t = f(Q^t)$ with

$$U_\nu^\mu(x) = f(\mathbf{X})_\nu^\mu(x), \quad G_\nu^\mu(x) = f(\mathbf{X} + \mathbf{A})_\nu^\mu(x),$$

where $f(\mathbf{Z})(x) = f(\mathbf{Z}(x))$ is the triangular matrix which is the function of the matrix $\mathbf{Z}(x)$, representing $\mathbf{Q}^{t(x)}(x)$ and $\mathbf{Q}^{t(x)]}(x)$ correspondingly as $\mathbf{X}(x) = \epsilon(\mathbf{Q}^{t(x)}(x))$ and

$$\mathbf{X}(x) + \mathbf{A}(x) \ , \quad A_\nu^\mu(x) = \epsilon(\dot{Q}^{t(x)]}(\mathbf{x}_\nu^\mu) - \dot{Q}^{t(x)}(\mathbf{x}_\nu^\mu)).$$

For the ordered functions $U^t = f(X_1^t, \dots, X_n^t)$ this can be written in terms of $\mathbf{X}_i$ with $\mathrm{d}\mathbf{X}_i = \mathrm{d}\Lambda(\mathbf{A}_i)$ and $\mathbf{Z}_i = \mathbf{X}_i + \mathbf{A}_i$ as

$$\mathrm{d}U = \mathrm{d}\Lambda(f(\mathbf{Z}_1, \dots, \mathbf{Z}_n) - f(\mathbf{X}_1, \dots, \mathbf{X}_n)) \ . \tag{3.2}$$

In particular, if all the triangular matrices $\{\mathbf{X}_i, \mathbf{Z}_i\}$ are commutative, then one can obtain the exponential function $U^t = \exp\{X^t\}$ for $X = \sum X_i$ as the solution of the following QS differential equation

$$\mathrm{d}U = \mathrm{d}\Lambda\left[(\exp\{\mathbf{A}\} - \hat{\mathbf{I}})\ \mathbf{U}\right] , \tag{3.3}$$

with $\mathbf{A} = \sum \mathbf{A}_i$ and the initial condition $U^0 = \hat{I}$. Now we shall study the problem of the solution of the general QS evolution equation

$$\mathrm{d}U = \mathrm{d}\Lambda(\left(\mathbf{S} - \hat{\mathbf{I}}\right)\ \mathbf{U}) = \mathrm{d}\Lambda\,(\mathbf{B}\ \mathbf{U}), \tag{3.4}$$

defined by a matrix-function $\mathbf{B}(x) = [B^\mu_\nu(x)]$ with noncommutative operator-values

$$B^0_0(x) \colon \mathcal{G} \otimes \mathcal{E}(x) \to \mathcal{G} \otimes \mathcal{E}(x), \quad B^-_+(x) \colon \mathcal{G} \to \mathcal{G}\ ,$$
$$B^0_+(x) \colon \mathcal{G} \to \mathcal{G} \otimes \mathcal{E}(x), \quad B^-_0(x) \colon \mathcal{G} \otimes \mathcal{E}(x) \to \mathcal{G}\ ,$$

and $B^\mu_\nu(x) = 0$, if $\mu = +$ or $\nu = -$. In the adapted case this equation can be written according to Corollary 1 as $\mathrm{d}U = \mathrm{d}\Lambda(\mathbf{B})U$, which shows that its solution with $U^0 = \hat{I}$ should be defined in some sense as a chronologically ordered exponent $U^t = \Gamma_{[0,t)}(\mathbf{B})$. In particular, if $B^\mu_\nu(x) = \hat{I} \otimes l^\mu_\nu(x)$, where $\mathbf{l} = [l^\mu_\nu]$ is a triangular QS-integrable matrix-function with $l^\mu_\nu = 0$, if $\mu = +$ or $\nu = -$,

$$l^0_0(x) \colon \mathcal{E}(x) \to \mathcal{E}(x),\ \|l^0_0\|^t_\infty < \infty;\ l^0_+(x) \in \mathcal{E}(x),\ l^-_0(x) \in \mathcal{E}^*(x),\ \|l\|^t_2 < \infty; \|l^-_+\|^t_1 < \infty,$$

then U^t is defined as $I \otimes \Gamma_{[0,t)}(\mathbf{l})$, where $\Gamma_{[0,t)}(\mathbf{l}) = \epsilon\left(\mathbf{f}^\otimes_{[0,t)}\right)$ is the representation (2.1) of $\mathbf{f}^\otimes_{[0,t)}(\boldsymbol{\varkappa}) = \otimes_{x\in\boldsymbol{\varkappa}}\mathbf{f}^t(x)$ with $\mathbf{f}^t(x) = \mathbf{1}(x) + \mathbf{l}^t(x)$, $\mathbf{l}^t(x) = \mathbf{l}(x)$ if $t(x) < t$ and $\mathbf{l}^t(x) = 0$, if $t(x) \geq t$, i.e. $\Gamma_{[0,t)}(\mathbf{l})$ is the second quantization $\Gamma(\mathbf{l}^t)$ of $\mathbf{l}^t$.

Theorem 3. *The QS evolution equation (3.4) written in the integral form $U^t = U^0 + \Lambda^t(\mathbf{BU})$ with $U^0 = \epsilon(T^0)$, $\mathbf{B}(x) = \epsilon(\mathbf{L}(x))$, is the representation ϵ of the recurrences*

$$T^{t+}(\boldsymbol{\varkappa}) = \left[F_{t(x)} \cdot T^t\right](\boldsymbol{\varkappa}), x \in \varkappa = \sqcup \varkappa^\mu_\nu\ , \tag{3.5}$$

defined for any partition $\boldsymbol{\varkappa} = (\varkappa^\mu_\nu)$ of a chain $\varkappa \in \mathcal{X}$ with $t \in (t_-(x), t(x)]$, $t_-(x) = \max\ t\left(\varkappa^{t(x)}\right)$ and $t_+ \in (t(x), t_+(x)]$, $t_+(x) = \min t(\varkappa_{t(x)})$, by

$$F_{t(x)}\left(\boldsymbol{\varkappa} \sqcup \mathbf{x}^\mu_\nu\right) = L^\mu_\nu(x, \boldsymbol{\varkappa}) + I\left(\boldsymbol{\varkappa} \sqcup \mathbf{x}^\mu_\nu\right) \equiv F^\mu_\nu(x, \boldsymbol{\varkappa}),$$

where $\mathbf{x}^\mu_\nu$ is one of the four single point tables $\boldsymbol{\vartheta} = (\vartheta^\kappa_\lambda)$ with $\vartheta^\mu_\nu = x$.

The recurrency (3.5) with the initial condition $T^t(\boldsymbol{\varkappa}) = T^0(\boldsymbol{\varkappa})$ for all $t \leq \min t(\varkappa)$ has the unique solution

$$T^t(\boldsymbol{\varkappa}) = [F_{[0,t)} \cdot T^0](\boldsymbol{\varkappa}), \quad F_{[0,t)} = \overleftarrow{\bullet}_{0\leq s<t} F_s\ , \tag{3.6}$$

where $F_s(\boldsymbol{\varkappa}) = I(\boldsymbol{\varkappa})$, if $s \notin \sqcup\varkappa^\mu_\nu = \varkappa$, defined for every chain $\varkappa = (x_1, \dots, x_m, \dots) \in \mathcal{X}$, $t \in (t_{m-1}, t_m]$ as product

$$\overleftarrow{\bullet}_{x\in\varkappa t} F_{t(x)} = F_{t_{m-1}} \cdots F_{t_1}$$

of $F_{t_i}, t_i = t(x_i) < t_{i+1}$ and T^0 in chronological order. The solution $U^t = \epsilon(T^t)$ of (3.4) is isometric $U^*U = \hat{I}$ (unitary: $U^* = U^{-1}$) up to a $t > 0$, if U^0 is isometric (unitary) and the triangular matrix-function $\mathbf{S}(x) = \mathbf{B}(x) + \hat{\mathbf{I}}(x) = \epsilon(\mathbf{F}(\varkappa))$ is pseudoisometric:

$$\mathbf{S}^\star(x)\mathbf{S}(x) = \hat{\mathbf{I}}(x) = \hat{I} \otimes \mathbf{1}(x)$$

(pseudounitary $\mathbf{S}^\star(x) = \mathbf{S}(x)^{-1}$) for almost all $x \in X^t$, that is

$$S_0^0(x)^* S_0^0(x) = I(x), \quad S_+^-(x)^* + S_+^0(x)^* S_+^0(x) + S_+^-(x) = 0 \tag{3.7}$$
$$S_+^-(x)^* + S_0^0(x)^* S_+^0(x) = 0, \quad S_+^0(x)^* S_0^0(x) + S_0^-(x) = 0$$

(and $S_0^0(x)$ is unitary $S_0^0(x)^* = S_0^0(x)^{-1}$ for almost all $x \in X^t$).

Proof. We are looking for the solution of the equation (3.4) as the representation $U = \epsilon(T)$ of some $T(\boldsymbol{\varkappa})$. If $\mathbf{B}(x) = \epsilon(\mathbf{L}(x))$, then $\mathbf{B}(x)\mathbf{U}(x) = \epsilon(\mathbf{L}(x)\mathbf{T}(x))$ and $\Lambda^t(\mathbf{BU}) = \epsilon(N^t(\mathbf{LT})$ due to the property $\Lambda \circ \epsilon = \epsilon \circ N$, proved in theorem 2, and the multiplicative property $\epsilon(\mathbf{LT}) = \mathbf{BU}$, where $\mathbf{U}(x) = \epsilon(\mathbf{T}(x))$, $\mathbf{T}(x)$ denotes the triangular matrix $[T_\nu^\mu(x)], T_\nu^\mu = 0$, if $\mu > \nu$, with $T_-^-(x) = T^{t(x)} = T_+^+(x)$ and $T_\nu^\mu(x) = \dot{T}^{t(x)}\ (\mathbf{x}_\nu^\mu)$ for $\mu \neq +$, $\nu \neq -$. This gives a possibility to consider the equation (3.4) in integral form as the representation $U^t = \epsilon(T^0 + N^t(\mathbf{LT}))$ of the equation

$$T^t(\boldsymbol{\varkappa}) = T^0(\boldsymbol{\varkappa}) + N^t(\mathbf{LT})(\boldsymbol{\varkappa}),$$

corresponding to $U^0 = \epsilon(T^0)$, where

$$N^t(\mathbf{C})(\boldsymbol{\varkappa}) = \sum_{\nu=0,+}^{\mu=-,0} \sum_{x \in \varkappa_\nu^\mu}^{t(x)<t} [\mathbf{L}(x)\mathbf{T}(x)]_\nu^\mu (\boldsymbol{\varkappa} \backslash \mathbf{x}_\nu^\mu)$$

depends on $T^s(\boldsymbol{\vartheta})$ with $s = t(x) < t$ for $x \in \sqcup \varkappa_\nu^\mu \supseteq \sqcup \vartheta_\nu^\mu$. This defines $T^t(\boldsymbol{\varkappa})$ for any partition $\boldsymbol{\varkappa} = (\varkappa_\nu^\mu)$ of a chain $\varkappa = (x_1, \dots, x_n) \in \mathcal{X}$ as the solution $T^t(\boldsymbol{\varkappa}) = T_m(\boldsymbol{\varkappa}), \qquad m = |\varkappa^t|$ of the recurrency

$$T_m(\boldsymbol{\varkappa}) = T_0(\boldsymbol{\varkappa}) + \sum_{k=1}^m \left[(F_{t_k} - I)\, T_{k-1}\right](\boldsymbol{\varkappa}) = \left[F_{t_m} T_{m-1}\right](\boldsymbol{\varkappa}),$$

where $T_{k-1} = T^{t_k}$ for $t_k = t(x_k)$, and the product $\mathbf{LT}$ for $T_\nu^\mu(x, \boldsymbol{\varkappa}) = T^{t(x)}(\boldsymbol{\varkappa} \sqcup \mathbf{x}_\nu^\mu)$ is written as

$$[\mathbf{L}(x)\mathbf{T}(x)]_\nu^\mu(\boldsymbol{\varkappa} \backslash \mathbf{x}_\nu^\mu) = \left[\left(F_{t(x)} - I\right) T^{t(x)}\right](\boldsymbol{\varkappa})$$

for $x \in \varkappa_\nu^\mu$ in terms of $F_{t(x)}(\boldsymbol{\varkappa} \sqcup x_\nu^\mu) = F_\nu^\mu(x, \boldsymbol{\varkappa})$ for $\mathbf{F} = \mathbf{L} + \mathbf{I}$. So, if the solution of (3.4) exists as $U^t = \epsilon(\mathbf{T}^t)$, then it is uniquely defined by (3.6).

Let us suppose, that $(\mathbf{S}^\star\mathbf{S})(x) = \hat{I} \otimes \mathbf{1}(\mathbf{x})$ for almost all x with $t(x) < t$, which is the representation $\epsilon(\mathbf{F}^\star\mathbf{F}) = \mathbf{S}^\star\mathbf{S}$ of $\left(F_{t(x)}^\star F_{t(x)}\right)(\boldsymbol{\varkappa} \sqcup \mathbf{x}) = I(\boldsymbol{\varkappa}) \otimes 1(\mathbf{x})$ for corresponding $\mathbf{F}(x)$, $t(x) < t$. By the recurrency $T_m = F_{t_m} T_{m-1}$ we obtain $(T_k^\star T_k)(\boldsymbol{\varkappa}) = I(\boldsymbol{\varkappa})$ for all $t_k < t$ from the initial condition $(T_0^\star T_0)(\boldsymbol{\varkappa}) = I(\boldsymbol{\varkappa})$. Hence, $(T^{t\star} T^t)(\boldsymbol{\varkappa}) = I(\boldsymbol{\varkappa})$ for almost all tables $\boldsymbol{\varkappa} = (\varkappa_\nu^\mu)$, namely those which are partitions of the chains $\varkappa$. This gives $U^{t*}U^t = \hat{I}$ for $U^t = \epsilon(T^t)$. In the same way one can obtain the condition $U^t U^{t*} = \hat{I}$ from $(\mathbf{SS}^\star)(x) = \hat{I} \otimes \mathbf{1}(x)$ for almost all x with $t(x) < t$. Writing the condition $\mathbf{S}^\star\mathbf{S} = \hat{I} \otimes \mathbf{1}$ in terms of matrix elements, we obtain (3.7):

$$(\mathbf{S}^\star\mathbf{S})(x) = \begin{bmatrix} 1 & S_+^0(x)^* & S_+^-(x)^* \\ 0 & S_0^0(x)^* & S_0^-(x)^* \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & S_0^-(x) & S_+^-(x) \\ 0 & S_0^0(x) & S_+^0(x) \\ 0 & 0 & 1 \end{bmatrix} = \hat{I} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & I(x) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The unitary solution of the equation (3.4) under conditions (3.7) in terms of $\mathbf{B} = \mathbf{S} - \hat{\mathbf{I}}$ was obtained in [1] in the framework of Itô (adapted) QS calculus for the stationary Markovian case $\mathbf{B}(t) = \mathbf{L} \otimes \hat{1}$, for nonstationary finite dimensional

Markovian case $\mathbf{B}(t) = \mathbf{L}(t) \otimes \hat{1}$ in [11]; and for non Markovian adapted case $\mathbf{B}(t) = \mathbf{L}^t \otimes 1_{[t}$ in [5]. ∎

Corollary 3. *Let $U^0 = T^0 \otimes \hat{1}, T^0 \in \mathcal{B}(\mathcal{H})$ and $\mathbf{S}(x) = \mathbf{F}(x) \otimes \hat{1}$ be defined by the operators $F^\mu_\nu = L^\mu_\nu,\ \mu < \nu,\ F^0_0 = L^0_0 + I$, acting as*

$$F^0_0(x) \colon \mathcal{H} \otimes \mathcal{E}(x) \to \mathcal{H} \otimes \mathcal{E}(x), \quad F^-_+ : \mathcal{H} \to \mathcal{H},$$
$$F^0_+(x) \colon \mathcal{H} \to \mathcal{H} \otimes \mathcal{E}(x), \quad F^-_0(x) \colon \mathcal{H} \otimes \mathcal{E}(x) \to \mathcal{H}$$

with

$$\|F^0_0\|^{(t)}_\infty < \infty,\ \|F^0_+\|^{(t)}_2 < \infty,\ \|F^-_0\|^{(t)}_2 < \infty,\ \|F^-_+\|^{(t)}_1 < \infty\ . \tag{3.8}$$

Then the solution $U^t = \epsilon\left(T^t\right)$ of the evolution equation (3.4) is defined as ζ-bounded operator for $\zeta > \|F^0_0\|^{(t)}_\infty$ by adapted chronological product $T^t(\boldsymbol{\varkappa}) = F_{[0,t)}(\boldsymbol{\varkappa}) \cdot T^0$, satisfying the recurrency

$$T^{t_+}(\boldsymbol{\varkappa} \sqcup \mathbf{x}) = \ F(\mathbf{x}) \cdot T^t(\boldsymbol{\varkappa}), \quad F(\mathbf{x}^\mu_\nu) = F^\mu_\nu(x)\ , \tag{3.9}$$

where $t_-(x) < t \le t(x) < t_+ \le t_+(x), t_-(x) = \max\{t < t(x) | t \in t(\varkappa)\}, t_+(x) = \min\{t > t(x) | t \in t(\varkappa)\}$, and $\cdot$ means the semitensor product, defined in theorem 1.

The QS process $U^t = \epsilon(T^t)$ is adapted, can be represented as multiple QS integral (1.5) $U^t = \Lambda_{[0,t)}(\mathbf{L}^\triangleleft \cdot T^0 \otimes \hat{1})$ with semitensor chronological products $\mathbf{L}^\triangleleft(\boldsymbol{\vartheta}) = \overleftarrow{\bullet}_{x \in \boldsymbol{\vartheta}} L(\mathbf{x})$ of $\mathbf{L}(x) = \mathbf{F}(x) - \mathbf{I}(x)$, and has the estimate

$$\|U^t\|^{\xi_-}_{\xi^+} \le \|T^0\| \exp\left\{\int_0^t (\|L^-_+(x)\| + (\|L^-_0(x)\|^2 + \|L^0_+(x)\|^2)/2\varepsilon\} \mathrm{d}x\right\} \tag{3.10}$$

for $\xi^+/\xi_- > \operatorname{ess\,sup}_{x \in X^t} \|F^0_0(x)\|$ and sufficiently small $\varepsilon > 0$.

Indeed, if $\mathbf{S} = \mathbf{F} \otimes \hat{1}$ and $\mathbf{F}$ satisfies the local integrability conditions (3.8), then

$$\| \overleftarrow{\bullet}_{x \in \boldsymbol{\varkappa}^t} F(\mathbf{x}) \| \le \prod_{x \in \boldsymbol{\varkappa}^t} \|F(\mathbf{x})\| = \prod_{\nu=0,+}^{\mu=-,0} \zeta^t(\varkappa^\mu_\nu)\ ,$$

where $\zeta^t(\varkappa^\mu_\nu) = \prod_{x \in \varkappa^\mu_\nu}^{t(x)<t} \|F^\mu_\nu(x)\|$. Hence, $T^t(\boldsymbol{\varkappa}) = F_{[0,t)}(\boldsymbol{\varkappa}^t) \otimes 1(\boldsymbol{\varkappa}_{[t})$ is relatively bounded $\|T^t\|(\boldsymbol{\zeta}^t) \le \|T^0\|$ with respect to $\boldsymbol{\zeta}^t(x) = (\|F^\mu_\nu(x)\|)^{\mu=-,0}_{\nu=0,+}$, $x \in X^t$, and $\boldsymbol{\zeta}^t(x) = 0$, $x \in X_{[t}$. Due to (2.4) and Theorem 2 this gives the ζ-boundedness of the operator $U^t = \epsilon(t_{[0,t)})$ with respect to $\zeta > \|F^0_0\|^{(t)}_\infty$; and the estimate (3.10) for $\xi^+ > \xi_- \|F^0_0\|^{(t)}_\infty, \varepsilon < \varepsilon(\xi^+, \xi_-)$ in terms of the norms (3.9) for $F^\mu_\nu = L^\mu_\nu + I \otimes 1^\mu_\nu$. Taking into account that $\epsilon \circ N_{[0,t)} = \Lambda_{[0,t)} \circ \epsilon$ and

$$N_{[0,t)}\left(\mathbf{L}^\triangleleft \cdot T^0\right)(\boldsymbol{\varkappa}) = \sum_{\boldsymbol{\vartheta} \subset \boldsymbol{\varkappa}^t} \mathbf{L}^\triangleleft(\boldsymbol{\vartheta}) \cdot T^0 \otimes 1(\boldsymbol{\varkappa} \backslash \boldsymbol{\vartheta}) = \left[\overleftarrow{\bullet}_{x \in \boldsymbol{\varkappa}^t} (I(\mathbf{x}) + L(\mathbf{x})) \cdot T^0\right] \otimes 1(\boldsymbol{\varkappa}_{t]})\ ,$$

we obtain the QS integral representation $\Lambda_{[0,t)}(\mathbf{L}^\triangleleft \cdot T^0 \otimes \hat{1})$ of Wick chronological product $\epsilon(T^t)$. This process is adapted and has the QS derivative

$$\mathbf{D}(x) = \Lambda_{[0,t(x))}(\mathbf{L}(x) \cdot \mathbf{L}^\triangleleft \cdot T^0 \otimes \hat{1}) = \mathbf{L}(x) \odot U^{t(x)}\ ,$$

where $(L \cdot U)^\mu_\nu = (L^\mu_\nu \otimes \hat{1}) \cdot U$, $B^\mu_0(x) \cdot U = B^\mu_0(U \otimes I(x))$, $B^\mu_+ \cdot U = B^\mu_+ U$. Hence multiple integral $U^t = \Lambda_{[0,t)}(\mathbf{L}^\triangleleft \otimes \hat{1})$ satisfies the QS equation (1.10) with $U^0 = \hat{I}$ as the case $\mathrm{d}U = \mathrm{d}\Lambda(\mathbf{L} \otimes \hat{1})U$ of (3.4).

Finally let us define the solution of the unitary evolution equation (1.10) with $\mathbf{L} = \mathrm{e}^{-\mathrm{i}\mathbf{H}} - \mathbf{I}$, $\mathbf{H}^\star = \mathbf{H}$, $U^0 = \hat{I}$ as the representation (2.1) of chronologically ordered products

$$T^t(\boldsymbol{\varkappa}) = \bullet^{\leftarrow}_{x\in\boldsymbol{\varkappa}} F^t(\mathbf{x}) = \mathbf{F}^{\triangleleft}(\boldsymbol{\varkappa}^t) \otimes 1(\boldsymbol{\varkappa}_{[t})$$

for $\mathbf{F}(x) = \exp\{-\mathrm{i}\mathbf{H}(x)\}$

$$\mathbf{F}^{\triangleleft}(\boldsymbol{\varkappa}) = F(\mathbf{x}_n)\cdots F(\mathbf{x}_1) \qquad \text{for} \qquad \boldsymbol{\varkappa} = \sqcup_{i=1}^n \mathbf{x}_i \ .$$

Here $F^t(\mathbf{x}) = I(\mathbf{x}) = I \otimes 1(\mathbf{x})$, if $t(x) \geq t$, $F^t(\mathbf{x}^\mu_\nu) = F^\mu_\nu(x)$, if $t < t(x)$, $\boldsymbol{\varkappa} = (\varkappa^\mu_\nu)$ is a partition $\varkappa = \sqcup^{\mu=-,0}_{\nu=0,+} \varkappa^\mu_\nu$ of a chain $\varkappa = (x_1, \dots, x_n) \in \mathcal{X}$ ordered by $t(x_{i-1}) < t(x_i)$ with $x_i \in \varkappa^\mu_\nu$, corresponding to the single point table $\mathbf{x}_i = (\varkappa^\mu_\nu)^{\mu=-,0}_{\nu=0,+}$ with $\varkappa^\mu_\nu = x_i$, and $F(\mathbf{x}) \cdot T(\boldsymbol{\vartheta})$ is the semitensor product,

$$F(\boldsymbol{\varkappa}) \cdot T(\boldsymbol{\vartheta}) = (F(\boldsymbol{\varkappa}) \otimes I^{\otimes}(\vartheta^0_0 \sqcup \vartheta^0_+))(T(\boldsymbol{\vartheta}) \otimes I^{\otimes}(\varkappa^-_0 \sqcup \varkappa^0_0)),$$

which is the usual product $F(\mathbf{x})T(\boldsymbol{\vartheta})$, if $\dim \mathcal{E} = 1$. As it follows from theorem 3, the solution $U^t = \epsilon\left(F_{[0,t)}\right)$ is unitary, if the triangular matrix-function $\mathbf{F}(x) = \exp\{-\mathrm{i}\mathbf{H}(x)\}$ is pseudo unitary, that is the Hamiltonian matrix-function $\mathbf{H} = [H^\mu_\nu]$ is pseudo Hermitian $H^\star(x) = H(x)$ for almost all $x \in X$:

$$H^{0*}_0 = H^0_0, H^{0*}_+ = H^-_0, H^{-*}_0 = H^0_+, H^{-*}_+ = H^-_+,$$

($H^\mu_\nu = 0$ for $\mu = +$, or $\nu = -$). One can easily find the powers $\mathbf{H}^n$ of the triangular matrix $\mathbf{H}$: $\mathbf{H}^0 = \mathbf{I}$, $\mathbf{H}^1 = \mathbf{H}$, $\mathbf{H}^2$ is defined by the table

$$\mathbf{H}^2 = \begin{pmatrix} H^-_0 H^0_0, & H^-_0 H^0_+ \\ H^0_0 H^0_0, & H^0_0 H^0_+ \end{pmatrix}, \ \mathbf{H}^{n+2} = \begin{pmatrix} H^-_0 H^{0n-1}_0, & H^-_0 H^{0n}_0 H^0_+ \\ H^{0n+2}_0, & H^{0n-1}_0 H^0_+ \end{pmatrix}, n = 1, 2, \dots$$

and $\mathbf{F} = \sum_{n=0}^\infty (-\mathrm{i}\mathbf{H})^{n/n!}$ as the triangular matrix $F^\mu_\nu = 0, \mu > \nu, F^-_- = 1 = F^+_+$,

$$\begin{aligned} F^0_0 &= e^{-\mathrm{i}H^0_0}, \quad F^-_+ = H^-_0 \left[\left(e^{-\mathrm{i}H^0_0} - I^0_0 + \mathrm{i}H^0_0\right)/H^0_0 H^0_0\right] H^0_+ - \mathrm{i}H^-_+ \\ F^0_+ &= \left[\left(e^{-\mathrm{i}H^0_0} - I^0_0\right)/H^0_0\right] H^0_+, \quad F^-_0 = H^-_0 \left[\left(e^{-\mathrm{i}H^0_0} - I^0_0\right)/H^0_0\right] \ . \end{aligned}$$

Representing the conjugated operators F^-_0, F^0_+ in the form $H^-_0 = F^* H^0_0 + \mathrm{i}E^*$, $H^0_+ = H^0_0 F - \mathrm{i}E$, where $E(x), F(x)$ are uniquely defined by $F^\star E = 0$, one can obtain the following canonical decomposition of the table (L^μ_ν) of the generating operators $L^\mu_\nu(x) = F^\mu_\nu(x) - I \otimes 1^\mu_\nu(x)$ of the unitary QS evolution U^t:

$$\begin{pmatrix} L^-_0 & L^-_+ \\ L^0_0 & L^0_+ \end{pmatrix} = \begin{pmatrix} F^* L^0_0 & F^* L^0_0 F \\ L^0_0 & L^0_0 F \end{pmatrix} + \begin{pmatrix} E^* & \frac{1}{2}E^* E \\ 0 & -E \end{pmatrix} + \begin{pmatrix} 0 & -\mathrm{i}H \\ 0 & 0 \end{pmatrix} \ ,$$

$$H = H^-_+ - F H^0_0 F^*, \qquad L^0_0 = \exp\{-\mathrm{i}H^0_0\} - I^0_0 \ .$$

Each of these three tables $\mathbf{L}_i$, $i = 1, 2, 3$ corresponds to a pseudounitary triangular matrix $\mathbf{F}_i = \mathbf{I} + \mathbf{L}_i$, satisfying the condition $\prod_{i=1}^3 \mathbf{F}_i = \mathbf{I} + \sum_{i=1}^3 \mathbf{L}_i = \mathbf{F}$ due to the orthogonality of $\mathbf{L}^i$. The first one can be diagonalized by the pseudounitary transformation $\mathbf{F}^\star_0 \mathbf{L}_1 \mathbf{F}_0 = \begin{pmatrix} 0 & 0 \\ L^0_0 & 0 \end{pmatrix}$. This defines the QS unitary evolution as the composition of three canonical types:

1) the Poissonian type evolution, given by the diagonal matrix-function $\mathbf{F}(x)$, corresponding to $H^\mu_\nu = 0$ for all $(\mu, \nu) \neq 0$, for which

$$U^t = \epsilon(F_{[0,t)}) = F^0_{0_{[0,t)}}, \qquad F^0_0(x) = \exp\{-\mathrm{i}H^0_0(x)\}$$

that is $U^t = \int^{\oplus} U^t(\varkappa)\mathrm{d}\varkappa$, where $U^t(\varkappa) = F_0^0(x_n)^t \cdots F_0^0(x_1)^t$ for any chain $\varkappa = (x_1, \dots, x_n) \in \mathcal{X}$, where $F_0^0(x)^t = F_0^0(x)$, if $t(x) < t$, otherwise $F_0^0(x)^t = I \otimes 1_0^0(x)$,

2) the quantum Brownian evolution, corresponding to $H_0^0 = 0 = H_+^-$ with $\mathrm{i}H_+^0 = E = \mathrm{i}H_0^{-*}$, and

3) Lebesgue type evolution, corresponding to $H_\nu^\mu = 0$ for all $(\mu, \nu) \neq (-, +)$ for which

$$U^t = \epsilon(F_{[0,t)}) = \hat{1} \otimes \int_{\mathcal{X}^t} (-\mathrm{i})^{|\varkappa|} \overleftarrow{\prod}_{x \in \varkappa} H_+^-(x)\mathrm{d}\varkappa = \overleftarrow{\exp}\left\{-\mathrm{i}\int_{X^t} H_+^-(x)\mathrm{d}x\right\} \otimes \hat{1} ,$$

where $\overleftarrow{\prod}_{x \in \varkappa} H(x) = H(x_n) \cdots H(x_1)$ is the usual chronological product of operators $H_+^-(x)$ in $\mathcal{H}$ defined for any chain $\varkappa = (x_1, \dots, x_n)$ by $t(x_{i-1}) < t(x_i)$. The sufficient conditions for the existence of the operators U^t as the representations ϵ of chronological products of the elements $F_\nu^\mu(x)$ of $\exp\{-\mathrm{i}\mathbf{H}(x)\}$ is the local QS integrability

$$\|H_0^0\|_\infty^t < \infty, \|H_+^0\|_2^t = \|H_0^-\|_2^t < \infty, \|H_+^-\|_1^t < \infty.$$

These conditions define the QS integral $\sum \Lambda_\nu^\mu(t, H_\nu^\mu) = \Lambda^t(\mathbf{H})$ as a (ξ^+, ξ_-)-continuous operator for any $\xi^+ > 1 > \xi_-$ and the QS time ordered exponential [14] $U^t = \overleftarrow{\exp}\{-\mathrm{i}\Lambda^t(\mathbf{H})\}$ as

$$U^t = \Gamma_{[0,t)}(\mathbf{L}) \equiv \epsilon\left(F_{[0,t)}\right)$$

even if $\mathbf{H}(x)$ is not pseudo Hermitian.

5. Non-Markovian QS processes and Langevin equations

Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a unital $*$-algebra of operators $A \in \mathcal{A}$, acting on a Hilbert space $\mathcal{H}$, and $j^t : \mathcal{A} \to \mathcal{B}(\mathcal{G})$ be a family of unital $*$-homomorphisms, representing $\mathcal{A}$ on $\mathcal{G} = \mathcal{H} \otimes \mathcal{F}$ as a QS process in the sense [12,13]:

$$j^t(A^*A) = j^t(A)^* j^t(A) \ , \ j^t(I) = \hat{I} \ .$$

We shall assume that each process $A^t = j^t(A)$ has a QS differential $\mathrm{d}j(A) = \mathrm{d}\Lambda(\boldsymbol{\partial}(A))$ in the sense

$$j^t(A) = j^0(A) + \sum \Lambda_\mu^\nu(t, \partial_\nu^\mu(A)) \ , \tag{4.1}$$

where $\boldsymbol{\partial} = (\partial_\nu^\mu)_{\nu=0,+}^{\mu=-,0}$ is a family of linear maps $\partial_\nu^\mu(x) : \mathcal{A} \to \mathcal{B}(\mathcal{G})$, depending on $x \in X$ in such a way that the table-function $\mathbf{D}(x) = \boldsymbol{\partial}(x, A)$ is QS integrable for every $A \in \mathcal{A}$. The QS derivative $\boldsymbol{\partial}$ of the process j was introduced by Hudson [14] in the Markovian case $\boldsymbol{\partial} = j \circ \boldsymbol{\lambda}$, corresponding to the assumption $\partial_\nu^\mu(x, \mathcal{A}) \subseteq j^{t(x)}(\mathcal{A})$ for all μ, ν and x. Using the adapted QS Itô formula he obtained the cohomology conditions for the maps λ_ν^μ: $\mathcal{A} \to \mathcal{A}$, which are necessary and sufficient in the constant case $\boldsymbol{\lambda}(x) = \boldsymbol{\lambda}$ for homomorphism property of j^t. These conditions can be written simply as unital $\star$-homomorphism property for $\boldsymbol{\varphi} = \boldsymbol{\lambda} + \mathbf{j}$, $\mathbf{j}(A) = A \otimes \mathbf{1}$

$$\boldsymbol{\varphi}(x, A^*A) = \boldsymbol{\varphi}(x, A)^\star \boldsymbol{\varphi}(x, A) \ , \ \boldsymbol{\varphi}(x, I) = \hat{I}$$

in terms of the linear maps $\boldsymbol{\varphi}(x) : A \to [\varphi_\nu^\mu(x, A)]$ into the triangular block-matrices $\mathbf{A} = [A_\nu^\mu] = \boldsymbol{\varphi}(A)$. In the scalar case $\mathcal{E}(x) = \mathbb{C}$

$$\varphi_\nu^\mu(A) = \lambda_\nu^\mu(A), \mu < \nu; \ \varphi_\nu^\mu(A) = \lambda_\nu^\mu(A) + A, \mu = \nu; \ \varphi_\nu^\mu(A) = 0, \mu > \nu,$$

is defined as the sum of the map $\boldsymbol{\lambda} = [\lambda^\mu_\nu]$ into the triangular matrices: $\lambda^\mu_\nu(A) = 0$ if $\mu = +$ or $\nu = -$ and the diagonal map $\mathbf{j} = [j^\mu_\nu]$, $j^\mu_\nu(A) = 0$, $\mu \neq \nu$, $j^\mu_\nu(A) = A$, if $\mu = \nu$. As an example one can consider the spatial $\star$-homomorphism

$$\boldsymbol{\varphi}(x, A) = \mathbf{F}^\star(x)(A \otimes \mathbf{1}(x))\mathbf{F}(x),$$

where $\mathbf{F}(x) = [F^\mu_\nu(x)]$ is a pseudounitary triangular matrix $F^\mu_\nu(x) = 0, \mu > \nu$ with $F^-_- = I = F^+_+$.

We shall prove as the consequence of the theorem 4 that the pseudo-homomorphism property of a locally QS integrable function $\boldsymbol{\varphi} = \{\boldsymbol{\varphi}(x)\}$ is also sufficient for the uniqueness and homomorphism property (4.1) of the solution j^t of QS Langevin equation (4.2) with a given initial $\star$-homomorphism j^0 in nonstationary and non-Markovian, and even in the nonadapted case.

Before doing this let us describe a decomposable operator representation $\boldsymbol{\mathcal{A}} = \int^\oplus \boldsymbol{\mathcal{A}}(\varkappa)\mathrm{d}\varkappa$ of the unital $\star$-algebra $\boldsymbol{\mathcal{A}}$ of relatively bounded $\mathcal{A}$-valued operator-functions $T(\boldsymbol{\varkappa})$ in a pseudo Hilbert space $\boldsymbol{\mathcal{G}} = \mathcal{H} \otimes \boldsymbol{\mathcal{F}}$. Here $\boldsymbol{\mathcal{F}}$ is the pseudo Fock space defined in [8,10] as usual Fock space over the space $\boldsymbol{\mathcal{K}} = L^1(X) \oplus \mathcal{K} \oplus L^\infty(X)$ of L^p-integrable vector-function $\mathbf{k}(x) = [k^\mu(x)|\ \mu = -, 0, +]$ with the pseudoscalar product

$$(\mathbf{k}|\mathbf{k}) = \langle k^-|k^+\rangle + \|k^0\|^2 + \langle k^+|k^-\rangle = \langle \mathbf{k}|\mathbf{g}\mathbf{k}\rangle,$$

$k^- \in L^1(X), k^+ \in L^\infty(X), k^0 \in \mathcal{K}$. In general case, when $\mathcal{K}$ is L^2-integral $\mathcal{K} = \int^\oplus \mathcal{E}(x)\mathrm{d}x$ of Euclidean (Hilbert) spaces $\mathcal{E}(x), x \in X$ with

$$k \in \mathcal{K} \Longleftrightarrow k(x) \in \mathcal{E}(x), \int \|k(x)\|^2 \mathrm{d}x < \infty,$$

$\boldsymbol{\mathcal{F}}$ consists of all integrable in the sense

$$\|\mathbf{k}\| = \sup_{\varkappa^+}[\int (\int \|k(\varkappa^-, \varkappa^0, \varkappa^+)\|\mathrm{d}\varkappa^-)^2 \mathrm{d}\varkappa^0]^{1/2} < \infty$$

tensor-functions $k(\varkappa^-, \varkappa^0, \varkappa^+) \in \mathcal{E}^\otimes(\varkappa^0) = \otimes_{x\in\varkappa^0}\mathcal{E}(x)$ of three chains $\varkappa^\mu \in \mathcal{X}$, $\mu = -, 0, +$ with the pseudoscalar product $(\mathbf{k}|\mathbf{k}) = \langle \mathbf{k}|\mathbf{g}^\otimes \mathbf{k}\rangle$,

$$(\mathbf{k}|\mathbf{k}) = \iiint \langle k(\varkappa^-, \varkappa^0, \varkappa^+)|k(\varkappa^+, \varkappa^0, \varkappa^-)\rangle \mathrm{d}\varkappa^- \mathrm{d}\varkappa^0 \mathrm{d}\varkappa^+, \tag{4.2}$$

Taking into account that $\bigcap \varkappa^\mu = \emptyset$ almost everywhere for the continuous measure $\mathrm{d}\varkappa$ on $\mathcal{X}$, and that for any $a \in \mathcal{H} \otimes \boldsymbol{\mathcal{F}}$

$$(a|a) = \int \sum_{\sqcup \varkappa^\mu = \varkappa} \langle a(\varkappa^-, \varkappa^0, \varkappa^+)|a(\varkappa^+, \varkappa^0, \varkappa^-)\rangle \mathrm{d}\varkappa \equiv (\mathbf{a}|\mathbf{a})$$

one can consider the space $\mathcal{G}$ as a pseudo Hilbert integral $\int^\oplus \mathcal{G}(\varkappa)\mathrm{d}\varkappa$ of tensor-functions $\mathbf{a}(\varkappa) \in \mathcal{H} \otimes \boldsymbol{\mathcal{E}}^\otimes(\varkappa) = \mathcal{G}(\varkappa), \boldsymbol{\mathcal{E}}(x) = \mathbb{C} \oplus \mathcal{E}(x) \oplus \mathbb{C}$ with values in direct sums $\mathbf{a}(\varkappa) = \oplus_{\sqcup\varkappa^\mu=\varkappa} a(\varkappa^-, \varkappa^0, \varkappa^+)$ over all the partitions $\varkappa = \varkappa^- \sqcup \varkappa^0 \sqcup \varkappa^+$ of $\varkappa \in \mathcal{X}$. The operator-valued functions $T(\boldsymbol{\varkappa})$ of $\boldsymbol{\varkappa} = (\varkappa^\mu_\nu)$ are unequally defined by decomposable operator $\mathbf{T} = \int^\oplus \mathbf{T}(\varkappa)\mathrm{d}\varkappa$ in $\boldsymbol{\mathcal{G}}$ acting as

$$[\mathbf{Ta}](\varkappa) = \sum_{\sqcup \varkappa^\mu = \varkappa} \oplus [Ta](\varkappa^-, \varkappa^0, \varkappa^+) = \mathbf{T}(\varkappa)\mathbf{a}(\varkappa),$$

$$[Ta](\varkappa^-, \varkappa^0, \varkappa^+) = \sum^{\mu=-,0,+}_{\sqcup_{\nu\geq\mu}\varkappa^\mu_\nu = \varkappa^\mu} T\begin{pmatrix} \varkappa^-_0 & \varkappa^-_+ \\ \varkappa^0_0 & \varkappa^0_+ \end{pmatrix} a(\varkappa^-_-, \varkappa^-_0 \sqcup \varkappa^0_0, \varkappa^-_+ \sqcup \varkappa^0_+ \sqcup \varkappa^+_+) \tag{4.3}$$

due to $\sqcup\varkappa^{\mu} = \sqcup\varkappa_{\nu}$ for $\varkappa^{\mu} = \sqcup_{\nu\geq\mu}\varkappa^{\mu}_{\nu}$, $\varkappa_{\nu} = \sqcup_{\mu\leq\nu}\varkappa^{\mu}_{\nu}$. It is easy to check that the pseudo conjugated operator $\mathbf{T}^{\star} = \int^{\oplus}\mathbf{T}(\varkappa)^{\star}\mathrm{d}\varkappa$ with respect to the pseudo scalar product (4.2) is also decomposable: $\mathbf{T}^{\star}(\varkappa) = \mathbf{T}(\varkappa)^{\star}$, and is defined as in (4.3), by $T^{\star}(\boldsymbol{\varkappa})$ and the product $(\mathbf{T}^{\star}\mathbf{T})(\varkappa) = \mathbf{T}^{\star}(\varkappa)\mathbf{T}(\varkappa)$ corresponds to the product (2.5). Moreover, the Fock representation (2.1) of the operator $\star$-algebra $\int^{\oplus}\boldsymbol{\mathcal{A}}(\varkappa)\mathrm{d}\varkappa$ can be described as a spatial transformation $\epsilon(T) = J^{\star}\mathbf{T}J$, where J is a pseudoisometric operator $(Ja|Ja) = \|a\|^2$, with $J^{\star}\colon \langle J^{\star}\mathbf{a}|a\rangle = (\mathbf{a}|Ja)$, acting as

$$[Ja](\varkappa^{-},\varkappa^{0},\varkappa^{+}) = \delta_{\emptyset}(\varkappa^{-})a(\varkappa^{0}), \qquad [J^{\star}\mathbf{a}](\varkappa) = \int a(\varkappa^{-},\varkappa,\emptyset)\mathrm{d}\varkappa^{-}\ ,$$

($\delta_{\emptyset}$ means the vacuum function $\delta_{\emptyset}(\varkappa) = 0$, if $\varkappa \neq \emptyset, \delta_{\emptyset}(\emptyset) = 1$). One can consider $J^{\star}\mathbf{T}J$ as a weak limit $t \to \infty$ of the operators $J^{\star}_{[0,t)}\mathbf{T}J_{[0,t)}$, well defined on $\mathcal{G}$ as $J_{[0,t)} = \int^{\oplus}_{\mathcal{X}^t} J(\varkappa)\mathrm{d}\varkappa$ due to

$$\|J_{[0,t)}a\|^2 = \iiint_{\varkappa^{\mu}\in[0,t)} \delta_{\emptyset}(\varkappa^{-})\|a(\varkappa^{0})\|^2\mathrm{d}\varkappa^{-}\mathrm{d}\varkappa^{0}\mathrm{d}\varkappa^{+} \leq\ \mathrm{e}^{t}\|a\|^2,$$

and to prove directly the property

$$J^{\star}\mathbf{T}^{\star}JJ^{\star}\mathbf{T}J = J^{\star}\mathbf{T}^{\star}\mathbf{T}J$$

corresponding to the multiplicativity property of ϵ.

Let $U^t = J^{\star}\mathbf{T}^tJ$ be the solution of the QS evolution equation (3.4) with $\mathbf{S} = \mathbf{J}^{\star}\mathbf{F}\mathbf{J}$, where $\mathbf{J} = J\otimes\mathbf{1}(x), 1^{\mu}_{\nu} = 0$, if $\mu \neq \nu$, $1^{-}_{-} = 1 = 1^{+}_{+}$, $1^{0}_{0}(x) = I(x)$ is the identity operator in $\mathcal{E}(x)$, and let $\boldsymbol{\tau}^t\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{A}}, \upsilon^t\colon \mathcal{B} \to \mathcal{B}$ be the corresponding transformations $\boldsymbol{\tau}(\mathbf{A}) = \mathbf{T}^{\star}\mathbf{A}\mathbf{T}$, $\upsilon(B) = U^{\star}BU$ of the algebras of relatively bounded operators respectively in $\boldsymbol{\mathcal{G}}$ and $\mathcal{G}$. Then one can obtain, denoting $\mathbf{E} = JJ^{\star}$,

$$J^{\star}(\boldsymbol{\tau}^t(A))J = J^{\star}\mathbf{T}^{t\star}\mathbf{A}\mathbf{T}^tJ = J^{\star}\mathbf{T}^{t\star}\mathbf{E}\mathbf{A}\mathbf{E}\mathbf{T}^tJ = \upsilon^t(J^{\star}\mathbf{A}J),$$

that is the QS process $\epsilon^t = \upsilon^t\circ\epsilon$ over the $\star$-algebra $\boldsymbol{\mathcal{A}}$ is the composition $\epsilon^t = \epsilon\circ\boldsymbol{\tau}^t$ of the representation ϵ and $\boldsymbol{\tau}^t = \int^{\oplus}\boldsymbol{\tau}^t(\varkappa)\mathrm{d}\varkappa$, where $\boldsymbol{\tau}(\varkappa,\mathbf{A}) = \mathbf{T}(\varkappa)^{\star}\mathbf{A}\mathbf{T}(\varkappa)$ is defined due to (3.6) as chronological compositions

$$\phi_{[0,t)}(\varkappa,\mathbf{A}) = \mathbf{F}^{\star}_{t_1}(\varkappa)\cdots\mathbf{F}^{\star}_{t_m}(\varkappa)\mathbf{A}\mathbf{F}_{t_m}(\varkappa)\cdots\mathbf{F}_{t_1}(\varkappa) = \left[\circ^{\rightarrow}_{x\in\varkappa^t}\boldsymbol{\phi}_{t(x)}(\varkappa)\right](\mathbf{A})$$

of the maps $\boldsymbol{\phi}_{t(x)}(\varkappa\sqcup x) = \boldsymbol{\phi}(x,\varkappa)$, and $\boldsymbol{\tau}^0(\varkappa,A) = \mathbf{T}^0(\varkappa)^{\star}\mathbf{A}\mathbf{T}^0(\varkappa)$: $\boldsymbol{\tau}(\varkappa) = \boldsymbol{\tau}^0(\varkappa)\circ\boldsymbol{\phi}_{[0,t)}(\varkappa)$, where

$$\boldsymbol{\phi}(\dot{\mathbf{A}}) = \int^{\oplus}\boldsymbol{\phi}(\varkappa,\dot{\mathbf{A}}(\varkappa))\mathrm{d}\varkappa = \mathbf{F}^{\star}\dot{\mathbf{A}}\mathbf{F}, \dot{\mathbf{A}}\in\dot{\boldsymbol{\mathcal{A}}} = \int^{\oplus}\int^{\oplus}\boldsymbol{\mathcal{A}}(\varkappa\sqcup x)\mathrm{d}\varkappa\mathrm{d}x.$$

Moreover, if a $\mathcal{B}$-valued process $B^t = \epsilon(\mathbf{A}^t)$ has a QS differential $\mathrm{d}B = \mathrm{d}\Lambda(\mathbf{D})$, then the transformed process $\hat{B}^t = \upsilon^t(B^t)$ satisfies the QS equation

$$\hat{B}^t = \hat{B}^0 + \Lambda^t(\hat{\boldsymbol{\sigma}}(\mathbf{G}) - \hat{\mathbf{B}}), \mathbf{G} = \mathbf{B} + \mathbf{D}\ , \tag{4.4}$$

where $\hat{\mathbf{B}}(x) = \mathbf{U}^{\star}(x)\mathbf{B}(x)\mathbf{U}(x) \equiv \boldsymbol{\upsilon}(x,\mathbf{B}(x))$, $\mathbf{B}(x) = \mathbf{J}^{\star}\dot{\mathbf{A}}^{t(x)}(x)\mathbf{J}$, $\boldsymbol{\sigma}(\mathbf{G}) = \mathbf{S}^{\star}\mathbf{G}\mathbf{S}$ as it fallows directly from (3.4) and the main formula (2.9):

$$\mathrm{d}\left(U^{\star}BU\right) = \mathrm{d}\Lambda\left(\mathbf{U}^{\star}\mathbf{S}^{\star}(\mathbf{B}+\mathbf{D})\mathbf{S}\mathbf{U} - \mathbf{U}^{\star}\mathbf{B}\mathbf{U}\right)\ .$$

In particular case $\mathbf{D} = 0$ this gives the QS Langevin (non adapted) equation for the QS process $\epsilon^t\colon \boldsymbol{\mathcal{A}} \to \mathcal{B}$, written in the differential form as

$$\mathrm{d}\epsilon^t(\mathbf{A}) = \mathrm{d}\Lambda^t(\boldsymbol{\epsilon}\circ\boldsymbol{\phi}(\dot{\mathbf{A}}) - \boldsymbol{\epsilon}(\dot{\mathbf{A}})) = \mathrm{d}\Lambda^t(\boldsymbol{\epsilon}\circ\boldsymbol{\lambda}(\dot{\mathbf{A}}))\ , \tag{4.5}$$

where $\dot{\mathbf{A}}(x) = \int^{\oplus} \mathbf{A}(x \cup \varkappa)\mathrm{d}\varkappa \in \dot{\boldsymbol{\mathcal{A}}}(x)$,

$$\boldsymbol{\lambda}(\dot{\mathbf{A}}) = \boldsymbol{\phi}(\dot{\mathbf{A}}) - \dot{\mathbf{A}}, \boldsymbol{\epsilon}(x, \dot{\mathbf{A}}) = \boldsymbol{\upsilon}(x, \mathbf{J}^{\star}\dot{\mathbf{A}}\mathbf{J}),$$

and $(\boldsymbol{\epsilon} \circ \boldsymbol{\phi})(\dot{\mathbf{A}}) = \boldsymbol{\upsilon} \circ \boldsymbol{\sigma}(\mathbf{J}^{\star}\dot{\mathbf{A}}\mathbf{J})$ due to

$$\mathbf{J}^{\star}\boldsymbol{\phi}(\dot{\mathbf{A}})\mathbf{J} = \mathbf{J}^{\star}\mathbf{F}^{\star}\dot{\mathbf{A}}\mathbf{F}\mathbf{J} = \mathbf{J}^{\star}\mathbf{F}^{\star}\mathbf{E}\dot{\mathbf{A}}\mathbf{E}\mathbf{F}\mathbf{J} = \mathbf{S}^{*}\mathbf{J}^{*}\dot{\mathbf{A}}^{*}\mathbf{J}\mathbf{S} = \boldsymbol{\sigma}(\mathbf{J}^{\star}\dot{\mathbf{A}}\mathbf{J}) \ .$$

The restriction of the equation (4.7) on the $\star$-subalgebra $\mathcal{A} \otimes \mathbf{1}^{\otimes} \in \boldsymbol{\mathcal{A}}, \mathbf{1}^{\otimes}(\varkappa) = \otimes_{x \in \varkappa}\mathbf{1}(x)$ gives the (non Markovian) Langevin equation (4.2) for $j^t = \epsilon^t \circ \mathbf{j}$, $\mathbf{j}(A) = A \otimes \mathbf{1}^{\otimes}$ with the QS derivative

$$\boldsymbol{\partial} = \boldsymbol{\epsilon} \circ \boldsymbol{\lambda} \circ \mathbf{j}, \quad \mathbf{j}(x, A) = A \otimes \mathbf{1}(x) \otimes \mathbf{1}^{\otimes}$$

over $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$.

Let us find the solution of the general Langevin QS equation (4.5) with nonspatial map $\boldsymbol{\varphi}$. It is given by the following theorem, which is an analog of the theorem 3 for the maps $\boldsymbol{\lambda}, \boldsymbol{\phi}, \boldsymbol{\tau}$ instead of the corresponding operators $\mathbf{L}, \mathbf{F}, \mathbf{T}$,.

Theorem 4. *The QS equation (4.5), written as $\epsilon^t = \epsilon^0 + \Lambda^t \circ \boldsymbol{\delta}$ for all $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ with*

$$\delta^{\mu}_{\nu}(x, \mathbf{A}) = \epsilon^{\mu}_{\nu}(x, \boldsymbol{\lambda}(x, \dot{\mathbf{A}})), \quad \epsilon^0(\mathbf{A}) = J^{\star}\boldsymbol{\tau}^0(\mathbf{A})J$$

is defined by linear decomposable maps $\boldsymbol{\lambda}(x) \colon \dot{\boldsymbol{\mathcal{A}}}(x) \to \dot{\boldsymbol{\mathcal{A}}}(x)$ with $\lambda^{\mu}_{\nu}(x, \mathbf{A}) = 0$, if $\mu = +$ or $\nu = -$ and $\boldsymbol{\tau}^0 \colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{A}}$ as the representation

$$\epsilon^t(\mathbf{A}) = J^{\star}\boldsymbol{\tau}^t(\mathbf{A})J \ , \ \boldsymbol{\epsilon}(x, \dot{\mathbf{A}}(x)) = \mathbf{J}^{\star}\dot{\boldsymbol{\tau}}^{t(x)}(x, \dot{\mathbf{A}}(x))\mathbf{J}$$

of the recurrences

$$\boldsymbol{\tau}^{t_+}(\varkappa) = \boldsymbol{\tau}^t(\varkappa) \circ \boldsymbol{\phi}_{t(x)}(\varkappa), \quad x \in \varkappa \in \mathcal{X} \ , \tag{4.6}$$

$t \in (t_-(\varkappa), t(x)]$, $t_+ \in (t(x), t_+(x)]$, where $t_{\pm} = t_{m\pm 1}$ for a chain $\varkappa = (x_1, \dots, x_m, \dots)$, $x = x_m$, $t_m = t(x_m)$, and

$$\boldsymbol{\phi}_{t(x)}(\varkappa \sqcup x, \mathbf{A}) = \boldsymbol{\lambda}(x, \varkappa, \mathbf{A}) + \mathbf{A} \equiv \boldsymbol{\phi}(x, \varkappa, \mathbf{A}), \mathbf{A} \in \boldsymbol{\mathcal{A}}(x \sqcup \varkappa) \ .$$

The recurrency (4.6) with initial condition $\boldsymbol{\tau}^t(\varkappa) = \boldsymbol{\tau}^0(\varkappa)$ for all $t \in [0, t_1]$ has the unique solution

$$\boldsymbol{\tau}^t(\varkappa) = \boldsymbol{\tau}^0(\varkappa) \circ \boldsymbol{\phi}_{[0,t)}(\varkappa) \ , \ \boldsymbol{\phi}_{[0,t)} = \circ^{\rightarrow}_{0 \le s < t}\boldsymbol{\phi}_s \tag{4.7}$$

defined for every $\varkappa = (x_1, \dots, x_m, \dots), t \in (t_{m-1}, t_m]$ by the chronological composition $\circ^{\rightarrow}_{x \in \varkappa^t}\boldsymbol{\phi}_{t(x)} = \boldsymbol{\phi}_{t_1} \circ \cdots \circ \boldsymbol{\phi}_{t_{m-1}}$ of $\boldsymbol{\phi}_{t(x)}(\varkappa) = \boldsymbol{\phi}(x, \varkappa \backslash x)$ for $x \in \varkappa^t = \{x \in \varkappa | t(x) < t\}$, $\boldsymbol{\phi}_s(\varkappa) = \mathbf{i}(\varkappa)$, $i(\varkappa)$ is the identity map $\boldsymbol{\mathcal{A}}(\varkappa) \to \boldsymbol{\mathcal{A}}(\varkappa)$, if $s \notin t(\varkappa)$. The solution $\{\epsilon^s\}$ of (4.5) is Hermitian: $\epsilon^s(\mathbf{A}^{\star}) = \epsilon^s(\mathbf{A})^*$ up to a $t > 0$, if $\boldsymbol{\tau}^0$ and $\boldsymbol{\phi}(x)$ are pseudo Hermitian:

$$\boldsymbol{\tau}^0(\mathbf{A}^{\star}) = \boldsymbol{\tau}^0(\mathbf{A})^{\star}, \boldsymbol{\phi}(x, \dot{\mathbf{A}}^{\star}(x)) = \boldsymbol{\phi}(x, \dot{\mathbf{A}}(x))^{\star}, x \in X^t,$$

and is (unital, faithful) QS-process, representing $\boldsymbol{\mathcal{A}}$, if $\boldsymbol{\tau}^0 \colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{A}}$ and $\boldsymbol{\phi}(x) \colon \dot{\boldsymbol{\mathcal{A}}} \to \dot{\boldsymbol{\mathcal{A}}}(x)$ are (unital) $\star$-endomorphisms (automorphisms) of the algebras $\boldsymbol{\mathcal{A}}$ and $\dot{\boldsymbol{\mathcal{A}}}(x)$ for almost all $x \in X^t$.

Proof. We look for the solution of the equation (4.5) as for the representation $\epsilon^t(\mathbf{A}) = J^{\star}\mathbf{A}^tJ$ of a process $\mathbf{A}^t = \boldsymbol{\tau}^t(\mathbf{A})$, transforming the $\star$-algebra $\boldsymbol{\mathcal{A}}$. From definition of $\boldsymbol{\epsilon}(x)$ and $\dot{\mathbf{A}}(x, \varkappa) = \mathbf{A}(\varkappa \sqcup x)$ we obtain

$$\boldsymbol{\epsilon}(x, \dot{\mathbf{A}}(x)) = \mathbf{J}^{\star}\hat{\mathbf{A}}(x)\mathbf{J}, \hat{\mathbf{A}}(x) = \dot{\boldsymbol{\tau}}^{t(x)}(x, \dot{\mathbf{A}}(x)) = \dot{\mathbf{A}}^{t(x)}(x),$$

and $\Lambda^t(\boldsymbol{\delta}(A)) = \Lambda^t(\mathbf{J}^\star\hat{\mathbf{L}}\mathbf{J}) = J^\star N^t(\hat{\mathbf{L}})J$, where $\hat{\mathbf{L}}(x) = \dot{\boldsymbol{\tau}}^{t(x)}(x,\dot{\mathbf{L}}(x)), \dot{\mathbf{L}}(x) = \boldsymbol{\lambda}(x,\dot{\mathbf{A}}(x))$. This gives the equation (4.6) in the integral form as the representation $J^\star(\dot{\mathbf{A}}^0 + N^t(\hat{\mathbf{L}}))J = J^\star\mathbf{A}^tJ$ of the equations

$$\boldsymbol{\tau}^0(\varkappa,\mathbf{A}(\varkappa)) + \sum_{x\in\varkappa^t}\dot{\boldsymbol{\tau}}^{t(x)}(x,\varkappa\backslash x,\boldsymbol{\lambda}(x,\varkappa\backslash x,\dot{\mathbf{A}}(x,\varkappa\backslash x))) =$$
$$\boldsymbol{\tau}^0(\varkappa,\mathbf{A}(\varkappa)) + \sum_{x\in\varkappa^t}\boldsymbol{\tau}^{t(x)}(\varkappa,\boldsymbol{\lambda}_{t(x)}(\varkappa,\mathbf{A}(\varkappa))) = \boldsymbol{\tau}^t(\varkappa,\mathbf{A}(\varkappa))$$

where $\boldsymbol{\lambda}_{t(x)}(\varkappa) = \boldsymbol{\lambda}(x,\varkappa\backslash x)$ for a $x\in\varkappa$. Denoting $\boldsymbol{\lambda}_t(\mathbf{A}) + \mathbf{A}$ as $\boldsymbol{\phi}_t(\mathbf{A})$, we obtain the recurrency (4.6) for $\boldsymbol{\tau}^t(\varkappa) = \boldsymbol{\tau}_m(\varkappa), m = |\varkappa^t|$ supposing the linearity of the maps $\boldsymbol{\tau}^t$:

$$\boldsymbol{\tau}_m(\varkappa) = \boldsymbol{\tau}_0(\varkappa) + \sum_{k=1}^m(\boldsymbol{\tau}_{k-1}(\varkappa)\circ\boldsymbol{\phi}_{t_k}(\varkappa) - \boldsymbol{\tau}_{k-1}(\varkappa)) = \boldsymbol{\tau}_{m-1}(\varkappa)\circ\boldsymbol{\phi}_{t_m}(\varkappa).$$

This recurrency has the unique solution (4.7), which is linear as the composition of the linear maps $\boldsymbol{\tau}^0$ and $\boldsymbol{\phi}_t$, what proves the uniqueness of the solution $\epsilon^t = \epsilon\circ\boldsymbol{\tau}^t$ of the equation (4.5).

If the maps $\boldsymbol{\tau}^0(\varkappa)$ and $\boldsymbol{\phi}(x,\varkappa) = \boldsymbol{\phi}_{t(x)}(\varkappa\sqcup x)$ are pseudo Hermitian, then the composition $\boldsymbol{\tau}^t(\varkappa)$ is also pseudo Hermitian, and if they satisfy the (unital) $\star$-endomorphism (automorphism) property

$$\boldsymbol{\tau}^0(\mathbf{A}^\star\mathbf{A}) = \boldsymbol{\tau}^0(\mathbf{A})^\star\boldsymbol{\tau}^0(\mathbf{A}), \boldsymbol{\phi}(x,\dot{\mathbf{A}}^\star\dot{\mathbf{A}}) = \boldsymbol{\phi}(x,\dot{\mathbf{A}})^\star\boldsymbol{\phi}(x,\mathbf{A})$$
$$(\boldsymbol{\tau}^0(\mathbf{A}^{-1}) = \boldsymbol{\tau}^0(\mathbf{A})^{-1},\ \boldsymbol{\phi}(x,\dot{A}^{-1}) = \boldsymbol{\phi}(x,\dot{A})^{-1})$$

for $x\in X^t$, then the compositions (4.7) have obviously the same properties. This proves the Hermiticity and (unital) homomorphism (isomorphism) property for the map $\epsilon^t\colon \mathbf{A}\in\boldsymbol{\mathcal{A}}\to J^\star\boldsymbol{\tau}^t(\mathbf{A})J$. ∎

Let us denote by $\boldsymbol{\mathcal{A}}^t\subseteq\boldsymbol{\mathcal{A}}$ the $\star$-subalgebra of relatively bounded operators $\mathbf{A} = \int^\oplus \mathbf{A}(\varkappa)\mathrm{d}\varkappa$ with $\mathbf{A}(\varkappa) = \mathbf{A}(\varkappa^t)\otimes\mathbf{1}^\otimes(\varkappa_{[t})$, and by $\mathcal{B}^t\subseteq\mathcal{B}$ the corresponding algebra of operators $B = B^t\otimes\hat{1}_{[t}$ with B^t, acting in $\mathcal{G}^t$. The adapted QS process ϵ^t over $\boldsymbol{\mathcal{A}}$ is defined by the condition $\epsilon^t(\boldsymbol{\mathcal{A}}^t)\subseteq\mathcal{B}^t$ for almost all t, and corresponds to the adapted QS evolution $\upsilon^t\colon\mathcal{B}\to\mathcal{B}, \upsilon^t(\mathcal{B}^t)\subseteq\mathcal{B}^t$, described as $\upsilon^t(B) = \epsilon^t(\mathbf{A})$ for $B = J^\star\mathbf{A}J$.

Corollary 4. *The QS process ϵ^t, defined by the equation (4.6), is adapted, if $\boldsymbol{\tau}^0(\boldsymbol{\mathcal{A}}^0)\subset\boldsymbol{\mathcal{A}}^0$ and $\boldsymbol{\phi}(x,\varkappa) = \boldsymbol{\phi}(x,\varkappa^{t(x)})\otimes\mathbf{i}(\varkappa_{[t(x)})$ for almost all $x\in X$. In that case the QS evolution υ^t is defined by adapted map $\boldsymbol{\sigma}(x)\colon \mathbf{J}^\star\dot{\boldsymbol{\mathcal{A}}}^{t(x)}(x)\mathbf{J}\to\mathbf{J}^\star\dot{\boldsymbol{\mathcal{A}}}^{t(x)}(x)\mathbf{J}$; the transformed adapted process $\hat{B}^t = \upsilon^t(B^t)$, with B^t, having the derivative $\mathbf{D}(x)\in\mathbf{J}^\star\dot{\boldsymbol{\mathcal{A}}}^{t(x)}(x)\mathbf{J}$, satisfies the QS differential equation*

$$\mathrm{d}\upsilon^t(B^t) = \upsilon^t[\mathrm{d}\Lambda^t(\boldsymbol{\sigma}(B\otimes\mathbf{1}) + \boldsymbol{\sigma}(\mathbf{D}) - B\otimes\mathbf{1})] \tag{4.8}$$

In particular, if $B^t = A\otimes\hat{1}$ and $\boldsymbol{\sigma}(x,A\otimes\hat{1}\otimes\mathbf{1}) = \boldsymbol{\varphi}(x,A)\otimes\hat{1}$, $\boldsymbol{\varphi}(x,\mathcal{A})\in\boldsymbol{\mathcal{A}}(x)$, where $\boldsymbol{\mathcal{A}}(x)$ is the algebra of $\mathcal{A}$-valued triangular matrices $\mathbf{A} = [A^\mu_\nu], A^\mu_\nu = 0$, if $\mu > \nu, A^-_- = A = A^+_+$ then the equation (4.8) has the form (4.1) in terms of $j^t(A) = \upsilon^t(A\otimes\hat{1})$, $\partial^\mu_\nu(x) = j^{t(x)}\circ\lambda^\mu_\nu(x)$, where

$$j^0(A) = \tau^0(A)\otimes\hat{1}, \boldsymbol{\lambda}(x,A) = \boldsymbol{\varphi}(x,A) - A\otimes\mathbf{1}(x), A\in\mathcal{A}.$$

If the maps $\varphi^\mu_\nu(x)$ are locally L^p-integrable in the sense

$$\|\lambda^0_0\|^t_\infty < \infty, \|\lambda^0_+\|^t_2 < \infty, \|\lambda^-_0\|^t_2 < \infty, \|\lambda^-_+\|^t_1 < \infty, \tag{4.9}$$

where $\|\lambda\|^t_p = \left(\int_{X^t} \sup_{A\in\mathcal{A}}\{\|\lambda(x,A)\|/\|A\|\}^p \mathrm{d}x\right)^{1/p}$, *then the solution* $j^t(A) = J^\star\boldsymbol{\tau}^0(\boldsymbol{\phi}_{[0,t)}(A\otimes\hat{1}))J$ *exists as relatively bounded QS integral*

$$j^t(A) = \Lambda_{[0,t)}(\boldsymbol{\tau}^0 \circ \boldsymbol{\lambda}^\triangleright(A)\otimes\hat{1}), \boldsymbol{\lambda}^\triangleright(\varkappa) = \circ^{\rightarrow}_{x\in\varkappa}\boldsymbol{\lambda}(x,\varkappa_{t(x)})$$

where $\boldsymbol{\lambda}(x,\varkappa) = \boldsymbol{\lambda}(x)\otimes\mathbf{i}^\otimes(\varkappa), \mathbf{i}^\otimes(\varkappa)$ *is the identity map for operators in* $\boldsymbol{\mathcal{E}}^\otimes(\varkappa)$ *and* $\boldsymbol{\tau}^0(\varkappa) = \tau^0\otimes\mathbf{i}^\otimes(\varkappa)$. *It has the estimate*

$$\|j^t(A)\|^{\xi_-}_{\xi^+} \le \|\tau^\circ\|\exp\{\int_{X^t}(\|\lambda^-_+(x)\| + (\|\lambda^0_+(x)\|^2 + \|\lambda^-_0(x)\|^2)/2\varepsilon)\mathrm{d}x\} \tag{4.10}$$

for $\xi^+/\xi_- > \operatorname{ess\,sup}_{x\in X^t}\|\varphi^0_0(x)\|, \|A\|\le 1$, *and sufficiently small* $\varepsilon > 0$.

Indeed, the process $\epsilon^t(A^t) = J^\star\boldsymbol{\tau}^t(A^t)J$ is adapted, iff $\boldsymbol{\tau}^t(\varkappa,\mathbf{A}) = \boldsymbol{\tau}^t(\varkappa^t,\mathbf{A}^t)\otimes\mathbf{1}^\otimes(\varkappa_{[t})$, as it was proven in the Corollary 2. But due to $\boldsymbol{\tau}(\varkappa,\mathbf{A}) = \boldsymbol{\tau}(\varkappa,\mathbf{A}(\varkappa))$, it is possible only in the case $\mathbf{A}^t(\varkappa) = \mathbf{A}^t(\varkappa^t)\otimes 1^\otimes(\varkappa_{[t})$ and $\boldsymbol{\tau}^t(\varkappa) = \boldsymbol{\tau}^t(\varkappa^t)\otimes\mathbf{1}^\otimes(\varkappa_{[t})$, what is equivalent to the corresponding conditions for $\boldsymbol{\tau}^0$ and $\boldsymbol{\phi}(x)$. If $B^t = J^\star\mathbf{A}^tJ$ is an adapted process: $\mathbf{A}^t(\varkappa) = \mathbf{A}^t(\varkappa^t)\otimes\mathbf{1}^\otimes(\varkappa_{[t})$, then

$$\dot{\mathbf{A}}^t(x,\varkappa) = \mathbf{A}^t(\varkappa\sqcup x) = \mathbf{A}^t(\varkappa)\otimes\mathbf{1}(x)\ , \qquad \forall t\le t(x)$$
$$\mathbf{B}(x) = \mathbf{J}^\star\dot{\mathbf{A}}^{t(x)}(x)\mathbf{J} = B^{t(x)}\otimes\mathbf{1}(x)\ , \qquad \forall x\in X$$

and $\hat{\mathbf{B}}(x) = \upsilon(x,\mathbf{B}(x)) = \hat{B}^{t(x)}\otimes\mathbf{1}(x)$, where $\hat{B}^t = \upsilon^t(B^t)$ for the transformed process $\boldsymbol{\upsilon}(x,B\otimes\mathbf{1}(x)) = \upsilon^{t(x)}(B)\otimes\mathbf{1}(x)$ evaluated in $B = B^{t(x)}$. This gives the equation (4.4) for the adapted process $\hat{B}^t$ in the differential form (4.8):

$$\mathrm{d}\upsilon^t(B^t) = \mathrm{d}\Lambda^t(\boldsymbol{\upsilon}(\boldsymbol{\sigma}(B\otimes\mathbf{1}+\mathbf{D}) - B\otimes\mathbf{1})) = \upsilon^t[\mathrm{d}\Lambda^t(\boldsymbol{\beta}(B)+\boldsymbol{\sigma}(\mathbf{D}))],$$

where $\boldsymbol{\beta}(B) = \boldsymbol{\sigma}(B\otimes\mathbf{1}) - B\otimes\mathbf{1}$ and $\mathrm{d}\Lambda^t\circ\boldsymbol{\upsilon} = \upsilon^t\circ\mathrm{d}\Lambda^t$ for the adapted evolution υ^t due to the same arguments, as in Corollary 1. This equation, restricted on $B^t = A\otimes\hat{1}$ has the integral form (4.1) due to $\mathbf{D} = 0$ where $j^t(A) = \upsilon^t(A\otimes\hat{1}), \boldsymbol{\partial}(x) = \upsilon^{t(x)}\circ\boldsymbol{\beta}(x)$ because $\boldsymbol{\upsilon}(x) = \upsilon^{t(x)}\otimes\mathbf{1}(x)$. If $\boldsymbol{\beta} = \boldsymbol{\lambda}\otimes\hat{1}$, where $\boldsymbol{\lambda}(x,A)\in\boldsymbol{\mathcal{A}}(x)$, then it can be written as

$$\mathrm{d}j^t(A) = \mathrm{d}\Lambda^t(j(\boldsymbol{\lambda}(A)) = j^t[\mathrm{d}\Lambda^t(\boldsymbol{\lambda}(A)\otimes\hat{1})]\ .$$

The solution $j^t(A) = J^\star\boldsymbol{\tau}^t(A\otimes\mathbf{1}^\otimes)J$ of this equation is defined by chronological composition (4.7) as

$$\boldsymbol{\tau}^t(\varkappa, A\otimes\mathbf{1}^\otimes) = \boldsymbol{\tau}^0(\varkappa,\boldsymbol{\varphi}^\triangleright(\varkappa^t,A)\otimes\mathbf{1}^\otimes(\varkappa_{[t})), \boldsymbol{\varphi}^\triangleright(\varkappa) = \circ^{\rightarrow}_{x\in\varkappa}\boldsymbol{\varphi}(x,\varkappa_{t(x)})\ ,$$

where $\boldsymbol{\tau}^0(\varkappa) = \tau^0\otimes\mathbf{i}\otimes(\varkappa)$ is an initial map, and $\boldsymbol{\varphi}(x,\varkappa) = \boldsymbol{\varphi}(x)\otimes\mathbf{i}(\varkappa)$. It can be described as $j^t(A) = \epsilon(T^t)$ by operator-valued function

$$T^t(\boldsymbol{\varkappa}) = \tau^0[\varphi(\mathbf{x}_1,\varphi(\mathbf{x}_2,\varphi(\mathbf{x}_m,A)))]\otimes\mathbf{1}^\otimes(\boldsymbol{\varkappa}_{[t})\ ,$$

$t_m < t \le t_{m+1}$, corresponding in (4.2) to the decomposable $\mathbf{T}^t = \boldsymbol{\tau}^t(A\otimes\mathbf{1}^\otimes)$ for a partition $\boldsymbol{\varkappa} = (\varkappa^\mu_\nu)$ of the chain $\varkappa = (x_1,\dots,x_m,\dots)\in\mathcal{X}$ with $\varphi(\mathbf{x}^\mu_\nu) = \varphi^\mu_\nu(x)$. Hence, the operator $\mathbf{T}^t$ is relatively bounded for $A\in\mathcal{A}$:

$$\|T^t(\boldsymbol{\varkappa})\| \le \|\tau^0\|\ \|A\|\prod_{x\in\boldsymbol{\varkappa}}\|\varphi^t(\mathbf{x})\| = \|\tau^0\|\ \|A\|\prod_{\nu=0,+}^{\mu=-,0}\zeta^t(\varkappa^\mu_\nu),$$

where $\boldsymbol{\varphi}^t(x) = \boldsymbol{\varphi}(x)$, if $t(x) < t$, otherwise $\boldsymbol{\varphi}^t(x) = \mathbf{i}(x)$, and $\zeta^t(\varkappa^\mu_\nu) = \prod_{x\in\varkappa^\mu_\nu}^{t(x)<t} \|\varphi^\mu_\nu(x)\|$. This proves like in the Corollary 3 the existence of the solution $j^t(A) = J^\star \mathbf{T}^t J = \epsilon(T^t)$ of the equation (4.1) as relatively bounded operator $B^t = j^t(A) \in \mathcal{B}$ for any $A \in \mathcal{A}$, having the estimate (4.10) for $\|A\| \leq 1$ in terms of $\|\varphi^\mu_\nu(x)\| = \sup\{\|\varphi^\mu_\nu(A)\|/\|A\|\}$, $\varphi^\mu_\nu = \lambda^\mu_\nu$ for $\mu < \nu$. It can be written in the form of the multiple QS integral (1.5) with respect to the operator function $B(\boldsymbol{\varkappa}) = L(\boldsymbol{\varkappa}) \otimes \hat{1}$ with

$$L(\boldsymbol{\varkappa}, A) = \tau^0[\lambda(\mathbf{x}_1, \lambda(\mathbf{x}_2, \dots, \lambda(\mathbf{x}_n, A)\dots))] = \tau^0[\lambda^\triangleright(\boldsymbol{\varkappa}, A)]$$

for any partition $\sqcup^{\mu=-,0}_{\nu=0,+}\varkappa^\mu_\nu = (x_1, \dots, x_n) \in \mathcal{X}$ where $\lambda(\mathbf{x}^\mu_\nu) = \lambda^\mu_\nu(x)$ on the single point table $\mathbf{x} = (\varkappa^\mu_\nu)$, $\varkappa^\mu_\nu = x$.

The solution of the nonstationary Markov Langevin equation in the form of multiple QS integral was obtained recently in the frame work of Itô QS calculus by Lindsay and Parthasarathy [15] for more restrictive conditions then (4.9) (finite dimensional and local bounded λ^μ_ν).

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6. References

(1) Hudson R.L., Parthasarathy K.R. *Quantum Itô's formula and stochastic evolution*, Comm. Math. Phys. **93**, (1984), 301-323.

(2) Meyer P.A., *Elements de Probabilités Quantiques*, Exposes I a IV, Institute de Mathematique, Université Louis Pasteur Strasbourg, 1985.

(3) Maassen H., *Quantum Markov processes in Fock space described by integral kernels*, in: "Quantum Probability and Applications II", ed. L. Accardi and W. von Waldenfels, Lecture Notes in Mathematics, 11–36, Berlin Heidelberg New York: Springer, 1985.

(4) Lindsay J.M., Maassen H., *An integral kernel approach to noise*, in: "Quantum probability and Applications III", ed. L. Accardi and W. von Waldenfels, Lecture Notes in Mathematics, 192–208, Berlin Heidelberg New York: Springer 1988.

(5) Accardi L., Quaegebeur J., *The Itô Algebra of Quantum Gaussian Fields*, Journal of Functional Analysis, **85**, N. 2 (1989), 213–263.

(6) Accardi L., Fagnola F., *Stochastic integration*, in: "Quantum Probability and Applications III", ed. L. Accardi and W. von Waldenfels, Lecture notes in Mathematics, 6–19, Berlin Heidelberg New York: Springer 1988.

(7) Belavkin V.P., *Nondemolition measurements, nonlinear filtering and dynamic programming of quantum stochastic processes*, in: "Modeling and Control of Systems", ed. A. Blaquiere, Lecture Notes in Control and Information Sciences, **121**, 245–265, Berlin Heidelberg, New York: Springer 1988.

(8) Belavkin V.P., *Nondemolition stochastic calculus in Fock space and nonlinear filtering and control in quantum systems*, in: "Stochastic methods in Mathematics and Physics", ed. R. Gielerak and W. Karwoski, 310–324, Singapour, New Jersey, London, Hong Kong: World Scientific, 1988.

(9) Belavkin V.P., *Quantum stochastic calculus and quantum nonlinear filtering*, Centro Matematico V. Volterra, Università degli Studi di Roma II, Preprint N. 6, Marzo, 1989.

(10) Belavkin V.P., *A new form and $\star$-algebraic structure of quantum stochastic integrals in Fock space*, Seminario matematico e fisico… (1989),…

(11) Holevo A.S., *Time-ordered exponentials in quantum stochastic calculus*, Preprint N. 517 Universität Heidelberg, June, 1989.

(12) Accardi L., Frigerio A., Lewis J.T., *Quantum stochastic processes*, publications RIMS **48**, (1982), 97–133.

(13) Belavkin V.P., *Reconstruction theorem for quantum stochastic processes*, Theor. Math. Phys., **62**, N. 3, (1985), 275–289.

(14) Hudson R.L., *Quantum diffusions and cohomology of algebras.* Proceedings First World congress of Bernoulli society, vol. 1, Yu. Prohorov, V.V. Sazonov (eds.), (1987), 479–485.

(15) Lindsay J.M., Parthasarathy K.R., *Cohomology of power sets with applications in quantum probability*, Comm. Math. Phys. **124**, (1989), 337–364.

(16) Malliavin P., *Stochastic calculus of variations and hypoelliptic operators* in: Itô K., (ed) Proc. of Int. Symp. Stoch. D. Eqs. Kyoto 1976 pp. 195–263. Tokyo: Kinokuniya–Wiley, 1978.

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